University of Heidelberg

Master Thesis

Spectral Networks - A story of Wall-Crossing in Geometry and Physics

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A thesis submitted in fulfilment of the requirements for the degree of Master of Science

in the

Faculty of Mathematics and Computer Sciences
Mathematical Institute

November 24, 2015

Declaration of Authorship

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Abstract

Faculty of Mathematics and Computer Sciences
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Master of Science

Spectral Networks - A story of Wall-Crossing in Geometry and Physics

by Sebastian SCHULZ

This thesis deals with the phenomenon of wall-crossing for BPS indices in $d = 4$ $\mathcal{N} = 2$ supersymmetric gauge theories with gauge group $\mathrm{SU}(K)$. Compactification over S^1 yields a three-dimensional σ-model with target space $\mathcal{M}$ a fiber bundle over the Coulomb branch $\mathcal{B}$ of the four-dimensional theory. We demonstrate how the wall-crossing is captured by smoothness conditions on the Hyperkähler metric of $\mathcal{M}$. Three ways of determining the $4d$ BPS spectrum are explained, drawing on the work of Gaiotto, Moore and Neitzke. Firstly, a twistor space construction reduces the problem to finding holomorphic Darboux coordinates which are obtained as solutions to a Riemann-Hilbert problem for large radii R of the circle. Secondly, for a subclass of theories obtained by compactifying a six-dimensional theory over a surface C, the Darboux coordinates can be computed from Fock-Goncharov coordinates on certain triangulations of C for gauge group $\mathrm{SU}(2)$. Thirdly, a codimension one sublocus of C called a Spectral Network captures the BPS degeneracies in a more efficient way.

Die vorliegende Abschlussarbeit beschäftigt sich mit dem Phänomen des Wall-Crossing für BPS Indizes vierdimensionaler Eichtheorien mit $\mathcal{N} = 2$ Supersymmetrie und Eichgruppe $\mathrm{SU}(K)$. Kompaktifizierung über einer S^1 liefert ein dreidimensionales σ-Modell, dessen Zielraum $\mathcal{M}$ ein Faserbündel über dem Coulombzweig $\mathcal{B}$ der vierdimensionalen Theorie ist. Wir zeigen auf, wie das Wall-Crossing geometrisch aufgefasst werden kann als Glattheit der Hyperkählermetrik von $\mathcal{M}$. Das Kernstück ist die Beschreibung dreier Wege das BPS Spektrum der 4d-Theorie zu bestimmen, was maßgeblich die Arbeit von Gaiotto, Moore und Neitzke war. Erstens kann durch eine Twistorraum-Konstruktion das Problem dazu reduziert werden, holomorphe Darboux-Koordinaten für das Faserbündel $\mathcal{M} \to \mathcal{B}$ zu finden. Diese lassen sich aus der Lösung eines Riemann-Hilbert-Problems bestimmen, die zumindest für große Radien des Kompaktifizierungskreises existiert. Zweitens beschreiben wir eine Unterklasse von Theorien, die ihrerseits als Kompaktifzierung einer sechsdimensionalen Theorie über einer gepunkteten Riemannschen Fläche entsteht. Die holomorphen Darboux-Koordinaten können für den SU(2)-Fall aus Fock-Goncharov-Koordinaten bestimmter Triangulierungen von C berechnet werden. Drittens und letztens erläutern wir, wie bestimmte Weben von Linien auf C, genannt Spektrale Netzwerke, die BPS Indizes sehr effizient kodieren.

Acknowledgements

I would like to thank my advisors Anna Wienhard and Daniel Roggenkamp for introducing me to this fascinating topic, along with their endless patience and the guidance I have received both academically and personally. My thanks go to the Differential Geometry Research Group for providing a warm and welcoming atmosphere. I am particularly grateful to Daniele Alessandrini for the helpful discussions.

I would like to thank my friends and family for helping me to keep my head where it belongs: Above this thesis. My particular thanks go to my friends Beatriz Tapia and Sebastian Nill for the careful proofreading and the positive criticism.

Contents

Chapter 1

Introduction

Supersymmetric gauge theories enjoy an intimate relationship to geometry, a fact that has led to a fruitful interplay of mathematics and physics. A particular interesting class of gauge theories are those in four dimensions and with $\mathcal{N} = 2$ supersymmetry. The physical content of this thesis deals with certain properties of these theories. The main factors contributing are the *Low Energy Effective Action* (LEEA) and their *BPS spectrum*, the spectrum of distinguished states, called *BPS states*, that are very stable. While the former can be computed explicitely through the acclaimed Seiberg-Witten theory, the latter have only been determined in very particular cases. We aim at describing different methods to derive this spectrum for a large class of theories, drawing on powerful methods from complex geometry.

Let us explain the structure of this thesis. We start in chapter 2 by explaining a functorial approach to Quantum Field Theories. Its spirit is to circumvent the direct notion of path integrals by indirectly encoding its properties in a (higher) functor, a method that has been put to great success for Topological and Conformal Field Theories. Our use for these methods are to describe two important concepts in modern theoretical physics: Compactification and defects. While the former is a technique to obtain a lower dimensional QFT from a higher dimensional one, the latter encode ways of embedding lower dimensional QFTs inside higher dimensional ones. The concepts of this chapter depend crucially on abstract notions from category theory which we have gathered in appendix A.

The main content of this thesis can be found in chapters 3-5.Each of these follow a similar pattern: The early sections introduce novel structures from physics through axiom systems that we aim to make as precise as possible. The mathematical implications are derived in the consecutive sections.

Chapter 3 introduces the notions of four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry required for further understanding. We have attached a primer to supersymmetry in appendix B which some readers might find useful. We review the notion of BPS states: Those are distinguished states of the theory which are very stable but can form or decay at certain walls in a manner governed by a wall-crossing formula. An index Ω is introduced in order to count these states. However, its computation can be infeasible due to the lack of proper tools in the strong coupling regime. A key insight is that Ω is locally constant over the moduli space of vacua $\mathcal{B}$ but jumps at the walls at which BPS states are generated or annihilated. Hence knowing the walls and how the index jumps, one can determine Ω globally by computing its value in a single region together with the jumps it is subject to. The question then remains how to determine these jumps.

We start in section 3.5 by presenting one way in which geometry can enter this picture. Upon compactification over a circle S^1_R of radius R, one obtains an $\mathcal{N} = 4$ theory in three-dimensions with moduli space a Hyperkähler manifold $\mathcal{M}$ in the IR limit. $\mathcal{M}$

is naturally a fiber bundle over $\mathcal{B}$ with generic fiber an abelian variety, but these fibers degenerate over the singular loci of $\mathcal{B}$. The wall-crossing formula is intimately related to the Hyperkähler metric g of $\mathcal{M}$: It is equivalent to saying that g is smooth. Hence, the problem of determining the indices Ω has been translated into computing the Hyperkähler metric g. We review a twistor space construction of this metric from [GMN10] in 3.6 in which holomorphic Darboux coordinates arise as solutions to a certain Riemann-Hilbert problem. The downside of this approach is that it only works for large radii R of the circle.

A different approach is possible for a subclass of theories that arise as compactification limits from a six-dimensional theory $\mathfrak{X}$. The theory $\mathfrak{X}$ is not very well-understood due to its lack of a proper action functional, hence compactifications are an important way to shed some light on it. A particular way of doing so is over a Riemann surface C with a positive number of punctures. The resulting four-dimensional theories enjoy $\mathcal{N} = 2$ supersymmetry (after a certain procedure called *partial topological twisting*) and are called *Theories of class* $\mathcal{S}$ ("$\mathcal{S}$" for "six"). Further compactification over the circle yields the same three-dimensional theory as before but with a novel structure. To see this, note that reversing the orienation of compactification must result in the same three-dimensional theory for physical reasons. Compactifying first to five dimensions yields (twisted) 5d super Yang-Mills theory. Further compactifying over C yields BPS equations which can be identified as the Hitchin equations. This way, the fiber bundle $\mathcal{M} \to \mathcal{B}$ becomes the Hitchin system.

$\mathcal{B}$ becomes the parameter space for the Higgs field φ and one can associate the spectral curve $\Sigma_u \subset T^*C$ to a base point $u \in \mathcal{B}$. Frequently abbreviating it by Σ, we can interpret it as a $K : 1$ branched cover over C when the gauge group is $\mathrm{SU}(K)$, where the sheets are labeled by the eigenvalues of φ. In physics, Σ is called the *Seiberg-Witten curve* that comes with the Seiberg-Witten differential λ, the restriction of the Liouville 1-form from T^*C to Σ. Introducing a local coordinate $w = \int_{z_0}^{z} \lambda$ on C gives us a new interpretation of BPS states: They are straight lines with inclination ϑ in the w-plane which are either closed or have both ends on a branch point. More generally, these straight lines give a foliation of C which we use in section 4.6 to define a triangulation of the surface for gauge group $\mathrm{SU}(2)$. Finally, touching the work of Fock and Goncharov [FG06] we obtain coordinates related to the triangulations which yield the Darboux coordinates needed to describe the Hyperkähler metric on $\mathcal{M}$. In a very elegant manner, the wall-crossing can be related to morphisms of the triangulations.

In chapter 5 we strive to explain the extension to higher rank $\mathrm{SU}(K)$ gauge groups. The key physical input are surface defects which at first glance make the story more complicated because we also consider certain two-dimensional theories and hence have to handle more data. After introducing certain functions, called Formal Parallel Transport F, and Spectral Networks, a web of WKB curves on C, we arrive at one of the main results, the Formal Parallel Transport Theorem 5.3.5. It states that assuming certain properties for these two that crucially include a 2d-4d version of a wall-crossing formula, the interplay between Spectal Networks and the functions F carries enough information to determine not only the 4d BPS spectrum but also the 2d solitons! This is both a vast simplification of the previous method and a powerful generalization that gives ground to many related applications.

We conclude with an outlook to both the variety of gaps we have left in our considerations and the possible directions that the methods explained here might lead to in the final chapter 6.

Chapter 2

Functorial Quantum Field Theories

A central role in studying Quantum Field Theories in physics is played by the path-integral, a tool that has proved to be very resistant to a rigorous mathematical definition due to an ill-defined measure on the space of field configurations. Atiyah's insight [Ati88] was to formalize properties of the path-integral without directly involving this measure. This approach has since been a fruitful mathematical subject called Topological Quantum Field Theory (TQFT), in which a TQFT is viewed as a (symmetric monoidal) functor from a bordism category of differentiable manifolds (possibly endowed with an orientation, a framing, etc.) into the category **Vect** of vector spaces. We explain this approach in more detail in section 2.1. Following this we describe a natural extension in which one does not only look at closed manifolds and bordisms between those, but also bordisms between these bordisms, bordisms between these bordisms of bordisms, and so forth. This structure is captured in the notion of higher categories: An extended TQFT is a higher functor between higher categories. A classification of these theories is given by the Cobordism Hypothesis: It states that one can reconstruct all the data of a fully extended TQFT by its value on a point. We roughly motivate and state these developments in section 2.2 before leaving the topological sector of QFT. The term "topological" means that the manifolds under consideration only depend on the diffeomorphism class. In section 2.3 we explain a possible generalization to manifolds endowed with geometrical data. The progress to extend these geometrical field theories into the language of higher categories has been very scarce, so we will need to assume some working properties that will become clearer as we move along. One advantage of the functorial point of view on QFTs is that a process called *compactification* becomes very natural. As we sketch in section 2.4, it is a way to obtain a lower dimensional QFT out of a higher dimensional one, a tool that is central for the content of the following chapters.

2.1 Classical TQFT's

The starting point for the discussion of TQFT's here are Atiyah's axioms [Ati88]. The definition of symmetric monoidal categories and functors are spelled out in appendix A.1, we will be interested in two main examples.

Example 2.1.1. Given a field k, the category $\mathbf{Vect}(k)$ of k-vector spaces forms a symmetric monoidal category with respect to the usual tensor product of vector spaces where the unit is given by the ground field k itself (considered as a vector space).

Example 2.1.2. Let $n \in \mathbb{N}_{>0}$. The category $\mathbf{Cob}(n)$ is given as follows:

- Objects in $\mathbf{Cob}(n)$ are represented by closed oriented $(n-1)$-dimensional manifolds.
- For two objects $M, N \in \mathbf{Cob}(n)$, a morphism B from M to N is represented by a bordism, i.e. an n-dimensional oriented manifold (with boundary) B together with a diffomorphism $\partial B \simeq \overline{M} \coprod N$ that preserves orientation. Here, $\overline{M}$ denotes the manifold M with the opposite orientation. Moreover, two such bordisms B, B' define the same morphism if there is a diffeomorphism $B \simeq B'$ extending the given diffeomorphism of boundaries $\partial B \simeq \overline{M} \coprod N \simeq \partial B'$.
- For an object $M \in \mathbf{Cob}(n)$, the identity morphism is induced by the "cylinder" bordism $B = M \times [0,1]$.
- Morphisms are composed by gluing the bordisms along their common boundary, i.e. given a triple of objects M, M', M'' and a pair of morphisms $B : M \to M', B' : M' \to M''$, they induce a morphism $\tilde{B} : M \to M''$ which is represented by the manifold $B \coprod_{M'} B'$ (which is indeed smooth and well-defined up to isomorphism).

The so defined category $\mathbf{Cob}(n)$ becomes symmetric monoidal under disjoint union of manifolds with the unit object given by the empty set (considered as an $(n-1)$-manifold).

Definition 2.1.3. (Atiyah) Let k be a field. An n-dimensional topological field theory is a symmetric monoidal functor $Z : \mathbf{Cob}(n) \to \mathbf{Vect}(k)$. More generally, given a symmetric monoidal category $\mathcal{C}$, we call a symmetric monoidal functor $Z : \mathbf{Cob}(n) \to \mathcal{C}$ a $\mathcal{C}$-valued topological field theory of dimension n.

Remark 2.1.4. Atiyah's definition means that an n-dimensional TQFT Z should assign the following data:

- To a closed oriented $(n-1)$-manifold M, a vector space $Z(M)$.
- A vector space homomorphism $Z(B) : Z(M) \to Z(N)$ is associated to an oriented bordism B with $\partial B = \overline{M} \coprod N$.
- A collection of isomorphisms $Z(\emptyset) \simeq k$ and $Z(M \coprod N) \simeq Z(M) \otimes Z(N)$ as well as a number of natural coherence properties which shall not be targeted here.

Before exploring lower-dimensional TQFTs, a couple general remarks are in order, which give important tools for making the TQFTs more explicit:

Remark 2.1.5. Given a closed oriented manifold B of dimension n, we can consider it as a bordism from the empty set to the empty set (seen as an $(n-1)$-dimensional manifold). An n-dimensional TQFT Z then yields a map $Z(B) : Z(\emptyset) \to Z(\emptyset)$ which can be further simplified using the canonical isomorphism $Z(\emptyset) \simeq k$ from the previous remark. Thus $Z(B) \in \mathrm{Hom}_{\mathbf{Vect}(k)}(k,k) \simeq k$ and an n-dimensional TQFT must hence assign to every closed oriented n-manifold B a number $Z(B) \in k$, the *partition function.*

Remark 2.1.6. Different ways of decomposing the boundary ∂B of an n-manifold B into two disjoint components yield different morphisms on the categorical level. Take for example a closed oriented $(n-1)$-manifold M disjointly united with its dual $\overline{M}$. Then there are 4 different ways to picture the cylinder $M \times [0,1]$ as a morphism, depending on the decomposition of its boundary:

1. It can be regarded as a bordism from M to itself, thus representing the identity morphism id_M in $\mathbf{Cob}(n)$.

2. It can be regarded as a bordism from $\overline{M}$ to itself, thus representing the identity morphism $\mathrm{id}_{\overline{M}}$ in $\mathbf{Cob}(n)$.

3. It can be regarded as a bordism from $M \coprod \overline{M}$ to the empty set. This corresponds to a morphism $M \coprod \overline{M} \to \emptyset$ which is called *evaluation map* ev_M.

4. It can be regarded as a bordism from the empty set to $M \coprod \overline{M}$. This corresponds to a morphism $\emptyset \to M \coprod \overline{M}$ which is called *coevaluation map* coev_M.

Now given an n-dimensional TQFT Z and a closed oriented $(n-1)$-dimensional manifold M, we can apply the functor Z to the evaluation map. This yields a canonical bilinear pairing

$$Z(M) \otimes Z(\overline{M}) \simeq Z(M \coprod \overline{M}) \longrightarrow Z(\emptyset) \simeq k. \tag{2.1.1}$$

This pairing is actually perfect, i.e. it induces an isomorphism $Z(\overline{M}) \xrightarrow{\sim} Z(M)^*$. In particular, Z(M) is a *finite dimensional* vector space ([Lur09]).
We can now put these results to use to explicitly describe TQFTs in dimensions one and two.

Example 2.1.7. (One-dimensional TQFTs)
Let Z be a one-dimensional TQFT. Objects on the left side are thus closed oriented zero-manifolds, i.e. a collection of points together with a choice of orientation for each point (up to orientation-preserving diffeomorphism). Denote an object by M, we can then decompose it as $M = M^+ \coprod M^-$ with M^+ (resp. M^-) consisting only of points with positive (resp. negative) orientation. Denote by $*^+$ (resp. $*^-$) the unique (up to orientation-preserving diffeomorphism) connected zero-manifold of positive (resp. negative) orientation. Since $\overline{*^+} \simeq *^-$, we find that $Z(*^+) = V$ and $Z(*^-) = V^*$ are finite-dimensional vector spaces. This completely determines the behaviour on the functor Z on *objects* of $\mathbf{Cob}(1)$ since the monoidality implies

$$Z(M) \simeq \left(\bigotimes_{p \in M^+} V \right) \otimes \left(\bigotimes_{p \in M^-} V^* \right).$$

To determine Z on *morphisms* of $\mathbf{Cob}(1)$, we need to look at oriented one-dimensional manifolds with boundary. Due to the monoidality of Z it suffices again to restrict to connected manifolds, i.e. to determine $Z(S^1)$ and $Z([0,1])$. For the latter we can make use of Remark 2.1.6 to find four possibilities, depending on the decomposition of the boundary:

1. $Z([0,1]) = \mathrm{id}_V$ for the bordism $[0,1] : *^+ \to *^+$.

2. $Z([0,1]) = \mathrm{id}_{V^*}$ for the bordism $[0,1] : *^- \to *^-$.

3. $Z([0,1]) = \mathrm{ev}_V : V \otimes V^* \to k : (v, \phi) \mapsto \phi(v)$ for the bordism $[0,1] : *^+ \coprod *^- \to \emptyset$.

4. $Z([0,1]) = \mathrm{coev}_V : k \to V \otimes V^* \simeq \mathrm{End}(V) : x \mapsto x \cdot \mathrm{id}_V$ for the bordism $[0,1] : \emptyset \to *^+ \coprod *^-$.

These morphisms are all determined by V. In order to describe the morphism $Z(S^1)$, we decompose the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ into two semi-circles $S^1_\pm = S^1 \cap \{z \in \mathbb{C} : \pm \mathrm{Im} z \geq 0\}$. Since gluing of bordisms corresponds to composition of maps, we see that

$Z(S^1)$ is giving as the composition

$$\begin{array}{ccccc} k \simeq Z(\emptyset) & \xrightarrow{Z(S^1_-)} & Z(\pm 1) \simeq V \otimes V^* & \xrightarrow{Z(S^1_+)} & Z(\emptyset) \simeq k \\ x & \longmapsto & x \cdot \mathrm{id}_V & \longmapsto & x \cdot \mathrm{tr}(\mathrm{id}_V) = x \cdot \dim(V), \end{array}$$

i.e. by multiplication with $\dim(V)$. To sum up, the entire one-dimensional TQFT Z is encoded in the value of Z at a single point $Z(*^+) = V$ (actually only the dimension of V is relevant since any two vector spaces of the same dimension are isomorphic) which is essentially the statement of the cobordism hypothesis in 1 dimension. All the other data which is *a priori* encoded in Z can be derived from this.

Example 2.1.8. (Two-dimensional TQFTs)
A similar procedure as for the former one-dimensional case can be applied in two dimensions. The starting point are closed, oriented manifolds of dimension 1. Due to monoidality it is possible to restrict the view to connected such manifolds, i.e. to S^1 (up to the choice of an orientation but there is an orientation-reversing diffeomorphism $S^1 \xrightarrow{\sim} \overline{S^1}$) which yields a vector space $V = Z(S^1)$ which in turn determines the value of Z on any *object* in $\mathbf{Cob}(2)$: any closed oriented 1-manifold M is a disjoint union of n circles, thus $Z(M) = V^{\otimes n}$.
Next, we want to figure out how the functor Z acts on *morphisms* in $\mathbf{Cob}(2)$, i.e. on bordisms of closed oriented manifolds. It is a well-known fact that each such surface can be obtained by gluing together pairs of pants and discs which leaves us with the task to find their image under Z:

1. The disc D^2 can be seen as a bordism $D^2 : S^1 \to \emptyset$ which yields a *trace map* $tr := Z(D^2) : V = Z(S^1) \to Z(\emptyset) \simeq k$
2. The pair of pants B can be seen as a bordism $B : S^1 \coprod S^1 \to S^1$ which yields a *multiplication map* $m := Z(B) : V \otimes V \to V$ which is commutative (under a flip of the incoming circles) and associative.
3. The gluing $B \coprod_{S^1} D^2$ gives a map by composition $V \otimes V \xrightarrow{m} V \xrightarrow{tr} k$ which corresponds to the image of the evaluation map ev_{S^1}, thus giving rise to a perfect pairing.

This pairing gives V the natural structure of a commutative Frobenius algebra, i.e. a commutative algebra over k with a linear map tr: $V \to k$ such that $(v, w) \mapsto tr(v \cdot w)$ is non-degenerate. Conversely, every finite-dimensional commutative Frobenius algebra V can be realized by a two-dimensional TQFT Z, i.e. $A = Z(S^1)$. This establishes an equivalence of categories between the category of two-dimensional TQFTs and the category of finite-dimensional commutative Frobenius algebras (for details see [Koc04]).

Even though this procedure seems to work very well thus far, it fails to carry over to higher dimensions where the topology becomes increasingly more difficult. A natural way to tackle higher dimensional TQFTs would be by looking at oriented n-manifolds M (recall such an M can always be seen as a cobordism $M : \emptyset \to \partial M$) and by trying a series of cutting and gluing actions to work with simpler pieces, such as simplices or a triangulation. In order to be able to do that, one would need to glue together not only along *closed* submanifolds of dimension $(n-1)$ but also along more general manifolds which may have a boundary themselves. This leads to the notion of manifolds with corners, and the definition of the TQFT has to be extended by working with *higher categories*.

2.2 Higher categories and the Cobordism Hypothesis

In order to define higher categories, we must first explain what it means to enrich a category. We will be rather sketchy but refer to [Gra83] for details. Let $\mathcal{M}$ be a monoidal category.

Definition 2.2.1. ($\mathcal{M}$-enriched category)
A (small) category $\mathcal{C}$ *enriched over* $\mathcal{M}$ is

- a set $\mathrm{ob}(\mathcal{C})$, called the set of objects;
- for each ordered tuple (a, b) of objects in $\mathcal{C}$, an object $C(a, b) \in \mathrm{ob}(\mathcal{M})$, called the *object of morphisms from* a *to* b;
- for each ordered triple (a, b, c) of objects, a morphism $\circ_{a,b,c} : C(b, c) \otimes C(a, b) \to C(a, c)$ in $\mathcal{M}$, called the *composition morphism*;
- for each object a, a morphism $j_a : I \to C(a, a)$, called the *identity element*,

such that composition is associative and unital.

An important example of a monoidal category is the category **Cat** of *small categories*. The product is given by the product category, as unit serves the category with only one object and only one morphism (the identity). This allows the following definition:

Definition 2.2.2. A *strict 2-category* is a category enriched over **Cat**.

In particular, in every strict 2-category the composition of 1-morphisms $a \xrightarrow{f} b \xrightarrow{g} c$ is strictly associative and composition with the identity morphisms strictly satisfies the identity law.
We want to formulate TQFTs in terms of (strict) 2-categories, thus we need to find analogues of $\mathbf{Cob}(n)$ and $\mathbf{Vect}(k)$. Since these are monoidal categories themselves, we may enrich categories over them. A category enriched over $\mathbf{Vect}(k)$ is called k*-linear category*. This leads to the notion of the strict 2-category $\mathbf{Vect}_2(k)$ which is given by the following data ([Lur09]):

- objects are *cocomplete* k-linear categories, i.e. k-linear categories which are closed under formation of direct sums and cokernels;
- for any pair of objects $\mathcal{C}, \mathcal{D}$ the category of morphisms $\mathrm{mor}_{\mathbf{Vect}_2(k)}(\mathcal{C}, \mathcal{D})$ is given by the category of *cocontinuous* k-linear functors, i.e. k-linear functors which preserve direct sums and cokernels;
- Composition and identity morphisms are given in the obvious way.

An example of a k-linear category is the category $\mathbf{Vect}(k)$ of k-vector spaces itself because the morphisms between two vector spaces V, W carry again the structure of a vector space via the identification $\mathrm{Hom}_{\mathbf{Vect}}(V, W) \cong V^* \otimes W$.

A similar approach to find a strict 2-category $\mathbf{Cob}_2(n)$ would be to take as objects closed oriented $(n-2)$-manifolds and for every two objects M, N to define the category $\mathcal{C} = \mathrm{mor}(M, N)$ of morphisms between them by taking bordisms from M to N as objects of $\mathcal{C}$ and diffeomorphism classes (relative to the boundary) of bordisms between these bordisms as morphisms of $\mathcal{C}$. We still need to describe the composition morphism in $\mathcal{C}$ and this is the real issue: it is hard to define an associative composition law. Since we do not consider the manifolds modulo diffeomorphism anymore, we face two problems:

1. The gluing itself is ill-defined because gluing along arbitrary codimension one submanifolds does not guarantee a smooth structure across the gluing. One must thus consider collar neighborhoods, which is a problem we postpone to section 2.3.

2. Even once the gluing produces smooth manifolds, the composition is not *strictly* associative, or else for objects M_1, M_2, M_3, M_4 and morphisms $C_i : M_i \to M_{i+1}$, $i = 1, 2, 3$, one would expect an equivalence

$$\left(C_1 \coprod C_2\right) \coprod C_3 = C_1 \coprod \left(C_2 \coprod C_3\right),$$

but in general there is only a canonical isomorphism.

One is thus forced to weaken the notion and consider *weak* 2-categories (also known as *bicategories*). As opposed to *strict* 2-categories, we do not require associativity (as in (A.1.1)) and unitality (as in (A.1.2)) *on the nose*. Rather, we require them to hold up to coherent 2-morphisms (for details on the coherence relations see [ML71, p. 281 ff]).

In order to establish the analogy to Atiyah's original definition, we have yet to view $\mathbf{Vect}_2(k)$ and $\mathbf{Cob}_2(n)$ as symmetric monoidal categories. In the case of $\mathbf{Cob}_2(n)$, the tensor product is obtained by disjoint union of manifolds (as it was already for $\mathbf{Cob}(n)$). For $\mathbf{Vect}_2(k)$ the case is more intricate, we refer to [Lur09] for details, a more general approach can be found e.g. in [TV08]. We are now able to define:

Definition 2.2.3. An n-dimensional 2-extended TQFT is a symmetric monoidal 2-functor $Z : \mathbf{Cob}_2(n) \to \mathbf{Vect}_2(k)$.

Here, by a 2-functor we mean what should be understood as a functor between weak 2-categories, i.e. a functor that preserves composition and identities up to coherent specified isomorphism (for details see again [ML71]).

Remark 2.2.4. This generalizes Atiyahs definition 2.1.3 in the following sense: given a 2-category $\mathcal{C}$, the morphisms between any two objects form a category. The canonical choice among those objects would be to extract the category of endomorphisms of the unit object $\Omega\mathcal{C} := \mathrm{mor}_{\mathcal{C}}(I, I)$. This way we obtain

$$\mathbf{Vect}(k) = \Omega\mathbf{Vect}_2(k) \text{ (for a field } k),$$
$$\mathbf{Cob}(n) = \Omega\mathbf{Cob}_2(n) \text{ (for a positive integer } n).$$

Accordingly, one can extract from a 2-extended TQFT $Z : \mathbf{Cob}_2(n) \to \mathbf{Vect}_2(k)$ a symmetric monoidal functor $\Omega Z : \mathbf{Cob}(n) \to \mathbf{Vect}(k)$, i.e. a TQFT in the sense of definition 2.1.3.

Naturally, a 2-extended TQFT carries more structure than the underlying TQFT. In particular, when cutting an n-dimensional manifold along a codimension 1 manifold, one is not restricted to closed cases but can consider the larger class of manifolds which themselves have closed submanifolds as boundaries. There is no natural reason to stop here though, one may want to cut along $(n-1)$-manifolds which have as boundary $(n-2)$-manifolds with boundary, which in turn might also have a boundary and so forth. Consequently, we can inductively define the notion of a *strict n-category* $\mathcal{C}$. Roughly speaking, that is a set of objects a, b, c such that for every pair of objects a, b the morphisms between them form an $(n-1)$-category $\mathrm{mor}_{\mathcal{C}}(a, b)$ which satisfy an associative and unital composition law. Similarly to the thought process for 2-categories, it is convenient not to demand associativity *on the nose* but rather associativity up to isomorphism which in turn has to be absorbed into the proper definition. Moreover, these isomorphisms must satisfy certain associativity conditions up to specified isomorphisms which need to be part of the definition again and so forth. This leads to the notion of a *weak n-category* or simply *n-category*. For our purposes it suffices to know that these coherence properties *can be* spelled out:

- for tricategories this has been carried out carefully in [Gur06] after the original definition by Gordon, Power and Street in [GPS95];
- for tetracategories this has been defined by Trimble [Tri06];

However, for weak n-categories with $n > 4$ this procedure is generally admitted to be infeasible which is why we will not dwell on this point but rather keep the intuitive picture in head and settle for giving a typical example. Let us just note that in the above way of counting 0-categories are sets and 1-categories are categories, while 2-categories coincide with our previous definition.

Example 2.2.5. Let $k \leq n$ be two non-negative integers. There is a k-category $\mathbf{Cob}_k(n)$ described (informally) as follows:

- objects are closed, oriented $(n-k)$-manifolds;
- a 1-morphism between two objects M, N is a bordism from M to N, i.e. an oriented $(n-k+1)$-manifold B together with a diffeomorphism $\partial B \xrightarrow{\sim} \overline{M} \coprod N$;
- 2-morphisms are assigned to every pair of objects M, N and pair of morphisms B, B' between them: it is represented by a bordism $P : B \to B'$ which is trivial along the boundary, i.e. there is a diffeomorphism

$$\partial P \xrightarrow{\sim} \overline{B} \coprod_{M \coprod \overline{N}} \left(\left(\overline{M} \coprod N \right) \times [0,1] \right) \coprod_{\overline{M} \coprod N} B';$$

$\vdots$

- a k-morphism is an oriented n-manifold X with corners (see e.g. [Joy12] for manifolds with corners) for which the structure of its boundary is determined by source and target of the corresponding bordism. We identify two n-manifolds with specified corners if there is an orientation-preserving diffeomorphism relative to their boundary between them.
- Composition of m-morphisms in $\mathbf{Cob}_k(n)$ $(1 \leq m \leq k)$ is given by gluing of the corresponding bordisms.

The composition is again associative up to (specified) diffeomorphism, giving $\mathbf{Cob}_k(n)$ the structure of a *weak* k-category. Disjoint union of manifolds makes it symmetric monoidal. For $k = 1$ this definition coincides with our previous definition 2.1.2 of $\mathbf{Cob}(n)$.

We will not dwell on finding the proper n-categorical generalization of $\mathbf{Vect}(k)$ since there are many different possibilities. Rather, we keep it very general by considering general symmetric monoidal n-categories:

Definition 2.2.6. Let $k \leq n$ be two non-negative integers and $\mathcal{C}$ be a symmetric monoidal k-category. A *k-extended n-dimensional $\mathcal{C}$-valued TQFT* is a symmetric monoidal k-functor $Z : \mathbf{Cob}_k(n) \to \mathcal{C}$.

Remark 2.2.7. We will be mostly interested in the case $k = n$. Such TQFT's are called *fully extended.*

Example 2.2.8. Let us take a look at some examples:

n=1 For the case $k = n = 1$ the notion of n-categories coincides with the notion of a category. Consequently, the definition 2.2.6 collapses to the notion of a 1-dimensional TQFT. These have been studied in example 2.1.7 and the punchline was that they are completely determined by their value on a point, i.e. by a finite-dimensional vector space.

n=2 The case $n = 2, k = 1$ has been studied in example 2.1.8 where we found that the entire TQFT is determined by its value on a circle, i.e. by a finite-dimensional commutative Frobenius algebra.
For $k = n = 2$ a TQFT can be evaluated on points and it has been shown in [SP09] that this evaluation in fact determines the entire TQFT: fully extended 2-dimensional TQFT's are equivalent to commutative separable finite-dimensional symmetric Frobenius algebras.

n=3 Things become increasingly more complicated and we do not attempt to make statements as precise as for the cases $n = 1$ and $n = 2$. Nonetheless, we want to briefly describe one important class of $3d$ TQFTs, namely Chern-Simons theories. These can be constructed from a compact connected simply-connected Lie group G and a cohomology class $c \in H^4(BG, \mathbb{Z})$ with certain additional restrictions, known as the *level* of the theory (here BG is the smooth moduli stack of G-principal connections). CS theories have been introduced by Witten in [Wit89], explaining the nature of the *Jones polynomial*, a then new invariant for 3-manifolds. In his language, CS theories were of type $n = 3, k = 1$ in our notation (though one should be careful because CS theories work with *unoriented* manifolds). A construction due to Reshetikhin-Turaev [RT91] extends this theory to $k = 2$ which assigns to S^1 a certain *modular tensor category* (these are braided tensor categories linear over a field which contain axioms for existence of duals, semi-simplicity with finitely many simples and a non-degeneracy condition, or, put even simpler, are "finite" in a very strong sense. For details see e.g. [FRS02]). Heuristically speaking, the data which the 2-functor assigns to *closed* manifolds should play the following role:

- A closed 3-manifold $\mapsto$ a complex number which is determined by a path integral computation over the *Chern-Simons action*,
- A closed 2-manifold $\mapsto$ a finite-dimensional vector space which is the space of sections of the Chern-Simons line bundle over the moduli space of flat connections on the surface (see e.g. [Bas09] for details on the Chern-Simons line bundle or, more generally, an introduction to CS theory),
- A closed 1-manifold (i.e. an S^1) $\mapsto$ a linear category which should be thought of as the category of level c positive energy representations of the loop group $\Omega_c(G)$.

The obvious question is if there is a way to extend CS theory to $k = 3$, i.e. make sense of its value on a point. There have been many attempts in this direction, most notably in the work of Freed, Hopkins, Lurie and Teleman [FHLT10] for abelian and finite groups and, more recently, by Henriques [Hen15]. Since they derive seemingly different answers to this question which haven't been related yet, we are not going to make their results precise here and close the discussion by simply stating that this question should still invoke interest, especially in the light of the Cobordism Hypothesis which we are going to state next.

As the examples in one and two dimensions have shown, the TQFT is completely determined by its value on a single point. This is exactly the claim of the Cobordism Hypothesis first noted by Baez and Dolan [BD95]. We state the theorem in the more modern version due to Lurie [Lur09]:

Theorem 2.2.9. (Baez-Dolan Cobordism Hypothesis)
Let $\mathcal{C}$ be a symmetric monoidal n-category. There is a bijective correspondence between isomorphism classes of framed extended $\mathcal{C}$-valued TQFTs and isomorphism classes of fully dualizable objects in $\mathcal{C}$ given by evaluation at a point $Z \mapsto Z(*)$.

Remark 2.2.10. We have yet to explain what a *framing* is and what it means for an object of $\mathcal{C}$ to be fully dualizable:

- A framing of an m-manifold M is a trivialization of its tangent bundle, i.e. a vector bundle isomorphism $T_M \xrightarrow{\sim} M \times \mathbb{R}^m =: \underline{\mathbb{R}}^m$. More generally, an *$n$-framing* for $n \geq m$ is a framing of the stabilized tangent bundle $T_M \oplus \underline{\mathbb{R}}^{n-m}$.

- Similarly to $\mathbf{Cob}_k(n)$ one can define the framed cobordism n-category $\mathbf{Cob}_k^{\mathrm{fr}}(n)$ where all manifolds have an n-framing (which induces isomorphisms of framings at the boundaries). This is indeed a symmetric monoidal k-category and gives rise to the notion of *framed $\mathcal{C}$-valued TQFTs* as symmetric monoidal k-functors $Z : \mathbf{Cob}_k^{\mathrm{fr}}(n) \to \mathcal{C}$. Indeed, every framing has an underlying orientation, so there is a forgetful functor $\mathbf{Cob}_k^{\mathrm{fr}}(n) \to \mathbf{Cob}_k(n)$ which determines a framed extended TQFT for every extended TQFT by composition.

- We do not want to dwell on making the notion of *fully dualizability* concrete. Note that by equation (2.1.1) we have seen that for n-dimensional TQFTs every closed oriented $(n-1)$-manifold determines a *finite* vector space due to a duality condition. The notion of fully dualizability is the higher categorical analogue of this, i.e. not only are the object and its morphisms demanded to have adjoints, but also all higher involved n-morphisms. This should be thought of as a generalization of the finiteness of vector spaces. A precise definition in the setting of (∞, n)-categories (which we will introduce momentarily) can be found in [Lur09, § 2.3]. A more detailed discussion (at least for $n = 2, 3$) was carried out in [SP09] and [DSPS13].

For the sake of completeness, we want to end this discussion with a vague explanation of the more general Cobordism Hypothesis due to Lurie which uses the notion of an (∞, n)-category. The basic observation is that n-categories are, as we have already seen, generally rather hard to describe, while n-groupoids (i.e. n-categories in which all morphisms are invertible) are much simpler to describe. A natural notion is then:

Definition 2.2.11. For a pair of non-negative integers $n \leq m$, an (m, n)-category is an m-category in which all k-morphisms are invertible for $n < k \leq m$.

Example 2.2.12. In particular, an (m, m)-category is the same as an m-category, while the notion of an $(m, 0)$-category coincides with that of an m-groupoid.

Consequently, an (∞, n)-category should be thought of as a "limit $m \to \infty$" of an (m, n)-category. For details we refer to [Ber11] and settle for an example.

Example 2.2.13. Recall the notion of the *fundamental n-groupoid* $\pi_{\leq n}X$ of a topological space X:

- objects are the points of X;
- for any two objects x, y, a 1-morphism in $\pi_{\leq n}X$ is given by a path from x to y in X;
- for a pair of objects x, y and a pair of 1-morphisms $f, g : x \to y$ between them, a 2-morphism from f to g in $\pi_{\leq n}X$ is given by a fixed endpoint homotopy of paths in X

⋮

- an n-morphism in $\pi_{\leq n}X$ is given by a homotopy between homotopies between ... between homotopies between paths between fixed points in X. We identify two such n-morphisms if the corresponding homotopies are homotopic (with fixed boundary);
- composition of 1-morphisms in $\pi_{\leq n}X$ is via composition of the corresponding paths;
- composition of k-morphisms ($k > 1$) in $\pi_{\leq n}X$ is via composition of the corresponding homotopies.

This determines indeed an n-category. Furthermore, reversal of the paths resp. homotopies inverts the corresponding morphisms (up to isomorphism), giving it the structure of an n-groupoid. The notion of $\pi_{\leq\infty}X$ is now straightforward: we simply keep adding homotopies between homotopies between ... and never stop this procedure. Of course, all of these homotopies can again be reversed which means that $\pi_{\leq\infty}X$ is an ∞-groupoid, i.e. an $(\infty, 0)$-category. As a matter of fact, the assignment $X \mapsto \pi_{\leq\infty}X$ determines a bijection between topological spaces (up to weak homotopy equivalence) and $(\infty, 0)$-categories ([Lur09]). This way, we can think of $(\infty, 0)$-categories as topological spaces.

One way to see (∞, n)-categories is then

Definition 2.2.14. (Lurie) Let $n > 0$. An (∞, n)-category $\mathcal{C}$ consists of the following data:

- a collection of objects;
- for any pair of objects $a, b \in \mathcal{C}$, an $(\infty, n-1)$-category $\mathrm{mor}_{\mathcal{C}}(X, Y)$ of 1-morphisms;
- a composition law for 1-morphisms which is associative and unital up to coherent isomorphism.

Example 2.2.15. An important example of an (∞, n)-category is $\mathbf{Bord}_n$ which is given by the following data:

- Objects of $\mathbf{Bord}_n$ are 0-manifolds;
- 1-morphisms of $\mathbf{Bord}_n$ are bordisms between 0-manifolds;
- 2-morphisms of $\mathbf{Bord}_n$ are bordisms between bordisms between 0-manifolds;

⋮

- n-morphisms of $\mathbf{Bord}_n$ are bordisms between bordisms between ... between 0-manifolds, i.e. n-manifolds with corners;
- $(n+1)$-morphisms in $\mathbf{Bord}_n$ are diffeomorphisms between the n-manifolds which are additionally required to reduce to the identity on the boundaries;
- $(n+2)$-morphisms in $\mathbf{Bord}_n$ are isotopies of diffeomorphisms;
- ....

As before, $\mathbf{Bord}_n$ obtains a symmetric monoidal structure by disjoint union of manifolds. Moreover, one can generally endow the considered manifolds with additional topological structure such as orientation or (n-)framing, giving rise to the corresponding symmetric monoidal (∞, n)-categories $\mathbf{Bord}_n^{\mathrm{or}}$ resp. $\mathbf{Bord}_n^{\mathrm{fr}}$.

Remark 2.2.16. A natural question at this point is that of the relationship between n-categories and (∞, n)-categories. Given an (∞, n)-category $\mathcal{D}$, we can extract an n-category $\mathrm{h}_n\mathcal{D}$, called the *homotopy n-category*, in the following way:

- objects of $\mathrm{h}_n\mathcal{D}$ are the objects of $\mathcal{D}$;
- for $k < n$, k-morphisms in $\mathrm{h}_n\mathcal{D}$ are the k-morphisms of $\mathcal{D}$;
- n-morphisms in $\mathrm{h}_n\mathcal{D}$ are the n-morphisms of $\mathcal{D}$ *up to isomorphism.*

Conversely, we can regard an n-category $\mathcal{C}$ as an (∞, n)-category in which the only k-morphisms for $k > n$ are the identity morphisms. This way, n-functors from $\mathrm{h}_n\mathcal{D}$ to $\mathcal{C}$ coincide with (∞, n)-functors from $\mathcal{D}$ to $\mathcal{C}$ (considered as an (∞, n)-category).
In particular, for $\mathcal{D} = \mathbf{Bord}_n^{\mathrm{fr}}$, $\mathrm{h}_n\mathcal{D}$ is given by $\mathbf{Cob}_n^{\mathrm{fr}}$. In this way of thinking, the following version of the Cobordism hypothesis due to Lurie is an extension of theorem 2.2.9:

Theorem 2.2.17. (Lurie Cobordism Hypothesis)
Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. There is a bijective correspondence between isomorphism classes of symmetric monoidal functors $Z : \mathbf{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$ and isomorphism classes of fully dualizable objects in $\mathcal{C}$ given by evaluation at a point $Z \mapsto Z(*)$.

2.3 Towards Functorial Quantum Field Theories

So far we have explored Topological Quantum Field Theories, a machinery that produces "mathematical structures" out of oriented, compact manifolds up to diffeomorphism. However, generic Quantum Field Theories will depend on additional data on the manifold X such as

- a symplectic structure on X,
- a principal G-bundle on X together with a flat connection,
- (a conformal class of) a (pseudo-)Riemannian metric on X (possibly with restrictions on the curvature),
- a supersymmetric structure,

to just name some common examples. The aim of Functorial Quantum Field Theory (or Geometric Field Theory) is to provide a mechanism that computes similar mathematical objects from an appropriate bordism category with the additional structure as a TQFT would do, i.e. we are looking for a (symmetric monoidal) functor into a proper (higher) category of vector spaces (or Hilbert spaces). In this section we will not dwell on taking the definitions into the spheres of higher category theory (this has only been done in a few exceptional cases) but will focus on reviewing the definitions of the enhanced bordism categories.
By the amount of different structures it becomes apparent that we are looking for a very flexible definition of the bordism category. The main issue arises in the gluing process: general geometric structures are not preserved in a smooth fashion by gluing along submanifolds of codimension 1 (note that we will not identify manifolds modulo diffeomorphism as we have done in the purely topological case), which makes it necessary to consider collars. We will follow the approach of Stolz and Teichner [ST11] who are mainly interested in Euclidean Field Theories (i.e. the manifolds considered in the bordism category are endowed with a flat Riemannian metric), however their construction easily generalizes to other geometries. Moreover, we will not attempt to fill in all details here but give a rather instructive and informal discussion. We will first explain how to add the new feature of collars to our previous definitions and then how to add the above mentioned variations.

Cobordisms with collars

Our goal here is to describe an alternative to the definition of $\mathbf{Cob}(n)$ from example 2.1.2. These will still be topological cobordisms, so we will call the category $\mathbf{TCob}(n)$ which is a category internal to **SymGrp**, the strict 2-category of symmetric monoidal groupoids (for the notion of internal categories and categories internal to strict 2-categories, see Appendix A.2 and A.3). In analogy to [ST11] we define (note that from now on manifolds will not be compact and will be without boundaries unless explicitly stated otherwise):

Definition 2.3.1. $\mathbf{TCob}(n)$ is defined as follows

- Objects in the *object groupoid* $\mathbf{TCob}(n)_0$ are quadruples $(M, M^c, M^{\pm})$ where M is an n-dimensional (smooth) manifold, M^c is a compact submanifold of codimension 1 called the *core* of M and there is a decomposition $M \backslash M^c = M^+ \coprod M^-$ into disjoint open submanifolds $M^{\pm}$ which contain M^c in their closure. The archetypical example would be for an arbitrary $(n-1)$-manifold X to take $M = X \times (-1, 1)$, $M^c = X \times \{0\}$, $M^+ = X \times (0, 1)$ and $M^- = X \times (-1, 0)$, i.e. M^c corresponds to an $(n-1)$-manifold with M a collar neighborhood. Note that we have absorbed orientation into the definition through the choice of $M^{\pm}$. We will mostly shorten the notation by suppressing M^c and $M^{\pm}$ and simply writing M for the quadruple. A morphism in the groupoid $\mathbf{TCob}(n)_0$ from M_0 to M_1 is the germ of a diffeomorphism $f : N_0 \to N_1$ of open neighborhoods $M_j^c \subset N_j \subset M_j$, such that
$$f(M_0^c) = M_1^c \text{ and } f(N_0^{\pm}) = N_1^{\pm} \tag{2.3.1}$$
for $N_j^{\pm} := N_j \cap M_j^{\pm}$ (i.e. f preserves the core and the "orientation"). Two such diffeomorphisms represent the same morphism in $\mathbf{TCob}(n)_0$ if they agree on smaller open neighborhoods of the core M_0^c. In particular, due to the tubular neighborhood theorem there is always a neighborhood N of M^c in M such that the pair (N, M^c) is diffeomorphic to the pair $(M^c \times (-1, 1), M^c \times \{0\})$.

- In the *morphism groupoid* $\mathbf{TCob}(n)_1$ objects consist of an ordered pair of objects $M_0, M_1 \in \mathbf{TCob}(n)_0$ and a *Topological bordism* $B = (B, i_0, i_1)$. Here, B is an n-dimensional manifold and i_j are smooth maps $i_j : N_j \to B$ where N_j are open neighborhoods of M_j^c in M_j as above. Letting $N_j^{\pm}$ be as above and $i_j^{\pm} := i_j \mid_{N_j^{\pm}}$ be the restrictions, the following two conditions are required to hold:

  1. i_j^+ are embeddings into $B \backslash i_j(N_j^- \cup M_j^c)$,
  2. the core $B^c := B \backslash \left(i_0(N_0^+) \cup i_1(N_1^-) \right)$ is compact.

  A *morphism* between bordisms B and B' in $\mathbf{TCob}(n)_1$ is the germ of a triple of diffeomorphisms $F : C \to C'$, $f_j : P_j \to P_j'$ $(j = 0, 1)$ where C (resp. P_j) is an open neighborhood of $B^c \subset B$ (resp. $M_j^c \subset N_j \cap i_j^{-1}(C)$) and analogously for C', P_j'. It is furthermore required that

  – f_j are morphisms in $\mathbf{TCob}(n)_0$ in the sense of (2.3.1),

  – the triple of diffeomorphisms has to be compatible with the structure of the Topological bordisms by making the following diagram commute:

$$\begin{array}{ccccc} P_0 & \xrightarrow{i_0} & C & \xleftarrow{i_1} & P_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle f_1} \\ P_0' & \xrightarrow{i_0'} & C' & \xleftarrow{i_1'} & P_1' \end{array} . \tag{2.3.2}$$

  As before with germs, two such triples of diffeomorphisms represent the same morphism in $\mathbf{TCob}(n)_1$ if they agree on smaller subsets in the obvious way.

- Lastly, we need to give the morphisms s, t, e and c from definition A.2.1:

  – The source and target functors $s, t : \mathbf{TCob}(n)_1 \to \mathbf{TCob}(n)_0$ assign to a bordism $B : M_0 \to M_1$ the source M_0 resp. target M_1.

  – The identity functor $e : \mathbf{TCob}(n)_0 \to \mathbf{TCob}(n)_1$ assigns to $M = (M, M^c, M^{\pm})$ the identity bordism id_M.

  – To define the composition $M_0 \xrightarrow{B_0} M_1 \xrightarrow{B_1} M_2$, remember that due to the definition of the bordisms there are maps $i_1 : N_1 \to B_0$ and $i_1' : N_1' \to B_1$. One can then glue along $N_1'' := N_1 \cap N_1'$ in a straightforward way to obtain a bordism $B_2 : M_0 \to M_2$. Note however, that the composition morphism c defined in this manner is not strictly associative but associative up to coherent isomorphism.

  These functors respect the symmetric monoidal structure of disjoint union of manifolds and are thus symmetric monoidal functors themselves.

Manifolds with rigid geometry

A very flexible notion of geometry is through the action of a group (thought of as an isometry group) on a model space in the spirit of Felix Klein and the Erlangen program. The easiest case would be for the model space to be a vector space.

Example 2.3.2. This is related to the following concept: Given a Lie group H together with an action on a finite-dimensional vector space V, an *H-structure* on a manifold X is an H-principal bundle P over X together with an isomorphism $P \times_H V \cong TX$. Such an H-structure is called *integrable* if it is locally flat. This notion includes:

- An integrable $GL_n(\mathbb{C})$-structure with $V = \mathbb{C}^n$ on a manifold X is the same as a complex structure.
- An integrable $O(n)$-structure with $V = \mathbb{R}^n$ on a manifold X is the same as a flat Riemannian metric.
- An integrable $U(n)$-structure with $V = \mathbb{C}^n$ on a manifold X is the same as a flat Kähler structure.
- An integrable $Sp(2n)$-structure with $V = \mathbb{R}^{2n}$ on a manifold X is the same as a symplectic structure.

By choosing charts on an open covering of X with codomain open subsets of V, one can identify an integrable H-structure with the lift of the derivatives of transition functions ϕ_{ij} along the map $\rho : H \to GL(V)$, such that the usual cocyle conditions are fulfilled (see (2.3.3)). Such an H-structure is then said to be *rigid* if every transition function ϕ_{ij} is the locally constant restriction of the action on V by the group $G = (V, +) \rtimes H$ of "translations and rotations".

However, the model space for a more general rigid geometry will be a smooth manifold rather than a vector space. So from now on, let $(G, \mathbb{M})$ denote a pair where G is a Lie group with an action on the manifold $\mathbb{M}$ (called the *model space*). The ideas explained here are very old, we follow along the lines of [ST11].

Definition 2.3.3. A $(G, \mathbb{M})$-*structure* on a manifold M consists of

1. an open covering $M = \bigcup U_i$, $U_i \subseteq M$ open;
2. a collection of diffeomorphisms (called *charts*) $\phi_i : U_i \xrightarrow{\cong} V_i \subseteq \mathbb{M}$ where $V_i \subseteq \mathbb{M}$ are open subsets in the model space $\mathbb{M}$;
3. a collection of elements $g_{ij} \in G$ which determine the transition functions by making the following diagram commute (here the g_{ij} are thought of as automorphisms on $\mathbb{M}$ via the action $G \times \mathbb{M} \to \mathbb{M}$)

$$\begin{array}{ccccc} & & \mathbb{M} & & \\ & \overset{\phi_j}{\nearrow} & & \overset{g_{jk}}{\nwarrow} & \\ U_i \cap U_j & & \Big\downarrow g_{ij} & & \mathbb{M} \\ & \underset{\phi_i}{\searrow} & & \underset{g_{ik}}{\swarrow} & \\ & & \mathbb{M} & & \end{array} \quad . \tag{2.3.3}$$

The equation $g_{ij} \cdot g_{jk} = g_{ik}$ expressed by the right triangle is called the *cocycle condition.*

In the spirit of manifolds with collars that we want to consider, this definition should be widened to state what a $(G, \mathbb{M})$-structure is on a pair (M, M^c):

Definition 2.3.4. Let $(G, \mathbb{M})$ be a geometry as above and $\mathbb{M}^c$ be a codimension one submanifold of $\mathbb{M}$. A geometry will from now on mean a triple $(G, \mathbb{M}, \mathbb{M}^c)$ but we will suppress $\mathbb{M}^c$ for convenience. A $(G, \mathbb{M})$-*structure on a pair* (M, M^c) consists of a maximal atlas (U_i, ϕ_i) for M as above, such that $\phi_i(U_i \cap M^c) \subseteq \mathbb{M}^c$.

Definition 2.3.5. The category $(G, \mathbb{M})$**-Man** of $(G, \mathbb{M})$-manifolds is given as follows:

- Objects are $(G, \mathbb{M})$-manifolds in the sense of definition 2.3.3;
- A morphism between two objects M, M' is a smooth map $f : M \to M'$ together with a choice $f_{i'i} \in G$ for each pair of charts (U_i, ϕ_i), $(U'_{i'}, \phi'_{i'})$ such that $f(U_i) \subseteq U'_{i'}$ making the following diagram commutative:

$$\begin{array}{ccc} U_i & \xrightarrow{\quad f \quad} & U'_{i'} \\ \Big\downarrow \phi_i & & \Big\downarrow \phi_{i'} \\ \mathbb{M} & \xrightarrow[\quad]{f_{i'i}} & \mathbb{M} \end{array},$$

and satisfying $f_{j'j} \cdot g_{ji} = g'_{j'i'} \cdot f_{i'i}$, i.e. commuting with the transition functions.

Remark 2.3.6. Given a $(G, \mathbb{M})$-manifold M, then any open subset $U \subset M$ inherits a $(G, \mathbb{M})$-structure and the map $U \to M$ is a morphism in $(G, \mathbb{M})$**-Man**. The same holds for any covering $N \to M$.

Remark 2.3.7. Similarly one can define the category of $(G, \mathbb{M})$-pairs. Then one needs to require that for a morphism $f : M_0 \to M_1$ cores are mapped to cores, i.e. $f(M_0^c) \subseteq M_1^c$.

Remark 2.3.8. Next we would like to give a definition of the bordism category of $(G, \mathbb{M})$-manifolds, $(G, \mathbb{M})$**-Cob**. This requires to first capture some additional technical features which shall not be covered here. In short, we need the following ingredients:

1. The notion of a symmetric monoidal category needs to be generalized to that of a symmetric monoid in a strict 2-category: Given a category $\mathcal{C}$ internal to a strict 2-category $\mathcal{A}$ such that C_0 is a terminal object in $\mathcal{A}$ (i.e. for every object X in $\mathcal{A}$ there exists only a single morphism $X \to C_0$), $\mathcal{C}$ is called a *monoid in* $\mathcal{A}$ (and the composition c is thought of as a multiplication with unit e). As an example, a monoid in the strict 2-category **Cat** of categories is a monoidal category (with C_0 the trivial category). If there is furthermore a *braiding* 2-isomorphism, one speaks of a *symmetric monoid* in $\mathcal{A}$ (see definition 2.16 in [ST11]). For example, a symmetric monoidal category is a symmetric monoid in **Cat**.

2. It is necessary to replace $(G, \mathbb{M})$**-Man** by its family version, i.e. all occuring manifolds come equipped with the extra datum of a smooth map (actually a submersion) into a parameter manifold S and the charts now have as codomain open subsets of $S \times \mathbb{M}$. The details for this are spelled out in definition 2.33 of [ST11]. To be more precise, one can introduce the notion of *fibered categories* (also called *Grothendieck fibrations*). Roughly speaking, a Grothendieck fibration is a functor $p : E \to B$ for which the fibers $E_b = p^{-1}(b)$ depend (contravariantly) functorially on $b \in B$. The details are spelled out in section 2.7 of [ST11]. A typical example is the functor $p :$ **Bun** $\to$ **Man** between the category of smooth fiber bundles and the category of manifolds, that sends a bundle to its base space. Similarly, the forgetful functor $(G, \mathbb{M})$**-Man** $\to$ **Man** that sends a $(G, \mathbb{M})$-family to its parameter space is a Grothendieck fibration. One should then more correctly write $(G, \mathbb{M})$**-Man/Man** for the category of families of $(G, \mathbb{M})$-manifolds to distinguish it from the non-family version. However, we will generally talk about the family version from now on and thus drop this distinction.

3. One needs to talk about *categories with flip.* This extra structure arises because any element $g \in Z(G)$ determines an automorphism $\theta_M := \theta_M(g) : M \to M$ for a $(G, \mathbb{M})$-manifold M by simply setting all $f_{i'i} = g$ in definition 2.3.5. Moreover, these *flips* θ_M form a natural family of isomorphisms and one then speaks of a category with flips. For two such categories $\mathcal{C}, \mathcal{D}$ with flips, a functor $F : \mathcal{C} \to \mathcal{D}$ between them is called *flip-preserving* if $F(\theta_M) = \theta_{FM}$. The notion of natural transformations is then automatic. Another example of a category with flip is the category **SVect** of super vector spaces with the flip given by the grading involution (see definition B.1.1). This is made more explicite in section 2.6 of [ST11].

We can now define the category $(G, \mathbb{M})$**-Cob** of cobordisms of $(G, \mathbb{M})$-pairs. The definition is similar to that of **TCob**(n), except that it is now a category internal to **Sym(Cat**$^{\mathrm{fl}}$**/Man)**, the strict 2-category of symmetric monoids in the 2-category of categories with flip fibered over **Man**, for the reasons explained above (note that the forgetful functor $(G, \mathbb{M})$**-Man** $\to$ **Man** preserves flips). Other than that, the definition is essentially the same by simply replacing the topological notions by their $(G, \mathbb{M})$-geometry analogues (see definition 2.46 in [ST11]).

One might at this point wonder what happens on the level of field theories. We still want a field theory to be a functor from this cobordism theory $(G, \mathbb{M})$**-Cob** into a proper category of vector spaces. This functor should however respect the given structures, i.e. it should be a functor between internal categories and its codomain should be a category **TV** internal to **Sym(Cat**$^{\mathrm{fl}}$**/Man)**. We skip a proper definition at this point (it can be found as definition 2.47 in [ST11]) and only note how the novel ingredients enter for the objects of $\mathbf{TV}_0$:

- The fibration over **Man** enters via taking as objects of $\mathbf{TV}_0$ a manifold S together with a certain *sheaf* of $\mathcal{O}_S$-modules (more precisely, complete, locally convex $\mathbb{Z}/2$-graded topological modules over the structure sheaf $\mathcal{O}_S$).
- The flip enters by the grading involution of this sheaf.
- The monoidal structure is given via the projective tensor product over $\mathcal{O}_S$.

Adding supersymmetry

Adding supersymmetry to the structures described above is then relatively easy because everything is expressed in terms of categories, thus we mainly need to find the right category. The first thing to do is replace the category **Man** of manifolds by the category **csM** of *complex super manifolds* (or **SMan** of real super manifolds, both notions are reviewed in appendix B). Concretely, one needs to make the following replacements:

- Lie groups (i.e. group objects in **Man**) need to be replaced by super Lie groups (i.e. by group objects in **csM**).
- Rigid geometries $(G, \mathbb{M})$ of a Lie group G acting on a model space $\mathbb{M}$ need to be replaced by their super versions: G a super Lie group acting on a supermanifold $\mathbb{M}$ of dimension $p|q$.
- $(G, \mathbb{M})$-manifolds M as in definition 2.3.3 naturally generalize to the super case (recall that topological properties need to be expressed in terms of the reduced manifolds). Similarly, for a submanifold $\mathbb{M}^c \subset \mathbb{M}$ of dimension $p - 1|q$ the notion of a $(G, \mathbb{M})$-structure on a pair generalizes definition 2.3.4.

- There is a category of $(G, \mathbb{M})$-supermanifolds. Moreover, one can take families of $(G, \mathbb{M})$-supermanifolds with a parameter supermanifold S, thus arriving at a Grothendieck fibration over **csM**.
- **csM** comes equipped with a flip: To every supermanifold $M = (|M|, \mathcal{O}_M)$ there is an automorphism

$$\begin{aligned} \theta_M : (|M|, \mathcal{O}_M) &\to (|M|, \mathcal{O}_M), \\ (x, f) &\mapsto (x, \pm f), \end{aligned}$$

  depending on whether f is even or odd.
- $(G, \mathbb{M})$-supermanifolds form a bordism category $(G, \mathbb{M})$**-Bord** that is defined in the same way as in the non-super case. Consequently, it is a category internal to **Sym(Cat**$^{\text{fl}}$**/csM)**, the strict 2-category of symmetric monoids in the 2-category of categories with flip fibered over **csM**. Here we require that there is an element g in the center of G, such that multiplication by g induces $\theta_{\mathbb{M}}$.

Similarly, one can generalize the notion of the codomain **TV** of the field theory functor (in particular, one uses now a supermanifold S for the parametrization of the family).

Definition 2.3.9. A *supersymmetric field theory* is a functor

$$Z : (G, \mathbb{M})\textbf{-Man} \to \textbf{TV}$$

internal to **Sym(Cat**$^{\text{fl}}$**/csM)**.

2.4 Compactification of QFTs

In this section we want to briefly illustrate an important tool of constructing "new" lower dimensional Quantum Field Theories out of existing ones. This section will not be mathematically rigorous because it demands higher category versions of geometrically enhanced bordism categories of which there is no rigorous definition to our best knowledge. Hence we will assume their existence in the following and settle for drawing only an instructive picture.

Let $0 \leq m \leq n$ be two integers and Z be an n-dimensional, m-extended quantum field theory, i.e. an m-functor from an m-category whose objects are represented by $(n - m)$-dimensional manifolds, possibly equipped with a geometry, whose 1-morphisms are represented by bordisms of those, whose 2-morphisms are represented by bordisms of bordisms etc. We use here a more general notion of geometry than in the previous section because we demand it to exist in arbitrary dimensions (or at least dimensions $\leq n$) such that the product of two manifolds with this geometry naturally carries this structure as well. A good example to think of is that of (Pseudo-)Riemannian manifolds.

Now fix a closed manifold K of dimension $k \leq m$ that carries the geometric structure that is under consideration. This defines an $(m - k)$-extended, $(n - k)$-dimensional quantum field theory which we will denote by Z/K and whose value on a manifold M of dimension $l \leq n - k$ is defined as

$$(Z/K)[M] := Z[K \times M]. \tag{2.4.1}$$

We will call this procedure *dimensional reduction*. It can be applied several times in a commutative way, i.e. for another closed manifold K' of dimension k' and with the necessary geometrical structure, there is again a quantum field theory

$$(Z/K)/K' = (Z/K')/K = Z/(K \times K'), \tag{2.4.2}$$

simply because the field theory functor is symmetric monoidal. However, the field theory obtained this way does not contain any new information but should rather be regarded as a restriction of the old field theory because it corresponds to evaluation of manifolds of the form $K \times M$ rather than more general manifolds. An honest lower dimensional field theory is obtained by *compactification*. For this to make sense, we need a geometry that allows a notion of "distance" on the manifolds, say by a metric or a conformal class thereof. Then in the same setting as above but with the manifold $K = (K, g)$ endowed with a metric, one obtains the field theory $Z/(K, g)$ the same way as above. The compactification of Z by K lacks a rigorous mathematical definition but is in physics thought of as a certain limit of functors

$$Z//K := \text{"} \lim_{\lambda \to 0} \text{"}\ Z/(K, \lambda \cdot g) \tag{2.4.3}$$

whose existence we must assume. In practice one expands the fields on the product manifold in Fourier modes of $(K, \lambda g)$. The limit is called "infrared limit" in physics because it corresponds to taking everything into account at low energies where the non-constant Fourier modes must be neglected due to their energy tending to infinity as $\lambda \to 0$. Hence, at low energies an effectively $(n-k)$-dimensional theory arises which is consequently called the *low-energy effective theory*. It is then a highly non-trivial statement that consecutive compactifications commute, i.e. that

$$(Z//K)//K' \overset{!}{=} (Z//K')//K. \tag{2.4.4}$$

Indeed, this is generally not true and if it should indeed hold, it usually has vast implications. As an example, the Geometric Langlands program can be described in this way as the consecutive compactification of a six-dimensional superconformal field theory by a torus and a Riemann surface [KW07]. We will exploit such an equality to find a close relationship between a class of four-dimensional theories ("theories of class $\mathcal{S}$") and Hitchin integrable systems in chapter 4.

We want to emphasize that even though this picture might be instructive, it is oversimplified. A true compactification is an elaborate and subtle process which we will witness in the next section when we compactify a four-dimensional theory to three dimensions.

Chapter 3

Four-dimensional $\mathcal{N} = 2$ Gauge theories

In this chapter we turn to actual physics. The theories at hand are four-dimensional gauge theories that enjoy $\mathcal{N} = 2$ supersymmetry and have been studied very thoroughly by physicists. The two things that we are mostly interested in are the Low Energy Effective Lagrangian (LEEA) and the BPS spectrum. From a more abstract perspective, this chapter may be read as a physical interpretation of the Kontsevich-Soibelman Wall-Crossing Formula (KSWCF) that describes the fusion of BPS particles.

The outline is the following: We start by briefly explaining two imporant examples of bosonic field theories in section 3.1, namely σ-models and gauge theory. Afterwards, we will turn to the representation theory of 4d $\mathcal{N} = 2$ gauge theories in section 3.2. We are particularly interested in BPS representations and will see that there are two types present which are called the hypermultiplet and the vectormultiplet. In section 3.3 we will develop an index that will assign integer numbers to representations of the super-Poincaré algebra. Its values are non-vanishing for BPS states but vanish on non-BPS states. The latter may be formed when two BPS states become aligned, hence forcing the BPS index to jump. This jumping behaviour may be described by the KSWCF which will play an important part in the rest of this thesis. In section 3.4 we recall the main features of Seiberg-Witten theory that has allowed vast improvements in the understanding of these theories. In particular, it determines the LEEA precisely, hence letting us focus on determining the BPS spectrum. Afterwards we will consider the four-dimensional theory on a space-time of the form $\mathbb{R}^3 \times S^1$ and reduce it over the second factor in 3.5. The resulting theory is a three-dimensional σ-model with target space a Hyperkähler manifold $(\mathcal{M}, g)$. We will see that by naive dimensional reduction we obtain a metric g^{sf} on $\mathcal{M}$ containing discontinuities that have to be resolved. This is done in 3.6 where we recall a twistor space construction of the Hyperkähler metric and see that the correction terms are related to the BPS indices: the jumps of these indices and the jumps of the "semiflat" metric g^{sf} have to be carefully balanced to obtain a continuous Hyperkähler metric g. The moral is that this balancing is captured by the KSWCF. This is the key insight of Gaiotto, Moore and Neitzke [GMN10] who give an explicit construction of g provided the radius R of the circle S^1 is large enough. In the following chapter we will present a different construction that holds for all R but is restricted to a smaller class of theories, called *Theories of class $\mathcal{S}$*.

3.1 Gauge theory and σ-models

We use this section to briefly review standard notions in physics for which we follow [DF99a]. We will omit supersymmetry in this section but the details are spelled out in [DF99b]. To start off, let us explain the concept of a *non-linear σ-model*. Fix two Riemannian manifolds M and X as well as a potential energy function $V : X \to \mathbb{R}$ which is usually required to be bounded from below, we will henceforth assume that it takes a minimum value of 0.

Definition 3.1.1. A *(non-linear) σ-model* is a quantum field theory for which the fields are maps $\phi : M \to X$ and for which the Lagrangian takes the form

$$L = \left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \phi^*V\right)\mu_g(x). \tag{3.1.1}$$

Here, g is the metric on M and μ_g is the associated density in local coordinate x.

An important quantity in physics is the *energy-momentum tensor*, here given locally by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + \left(-\frac{1}{2}|\mathrm{d}\phi|^2 + \phi^*V\right)g_{\mu\nu}.$$

The moduli space of vacua is defined to be the moduli space of field configurations minimizing the energy density. In case $M = \mathbb{R}^{1,3}$ this means minimizing T_{00} and hence the moduli space of vacua is given by

$$\mathcal{M}_{\text{vac}} := V^{-1}(0). \tag{3.1.2}$$

A second important class of quantum field theories are *gauge theories*. Let G be a Lie group (called the *gauge group*), M be a smooth manifold and $P \to M$ be a principal G-bundle over M with the G-action R_g from the right. Let V be a space with a left G-action, one can form the associated bundle $V^P = P \times_G V \to M$ with sections being equivariant maps $f : P \to V$. In the special case of $V = \mathfrak{g} = \mathrm{Lie}G$ together with the adjoint action, one obtains the adjoint bundle $\mathrm{ad}P := \mathfrak{g}^P$. Similarly, one obtains an adjoint bundle of groups $P \times_G G \to M$ whose sections are called *gauge transformations* and act as automorphisms on P.

For any smooth fiber bundle $\pi : E \to B$ one can define the *Ehresmann connection*: Let $V := \ker(\mathrm{d}\pi : TE \to TB)$ be the vertical bundle, an Ehresmann connection is a smooth subbundle H of TE such that $TE = H \oplus V$. This induces the *connection form* c, a vector bundle endomorphism on TE that is the projection onto V.

In the case at hand of a principal G-bundle $P \to M$, a principal Ehresmann connection (or simply connection) is an Ehresmann connection that is G-invariant, hence horizontally lifting any vector field η on M to a G-invariant vector field $\tilde{\eta}$ on P. Note that the differential of the vertical action of G on P allows for the identification of $\mathfrak{g}$ with the subspace V_x ($x \in M$) via a map $\iota : V_x \to \mathfrak{g}$. One can equivalently define a connection on P via a $\mathfrak{g}$-valued 1-form $A \in \Omega^1_P(\mathfrak{g})$ satisfying

$$\begin{aligned} \iota(X_V) &= A(X_V), X_V \in V, \\ R_g^*A &= \mathrm{Ad}_{g^{-1}}(A), g \in G. \end{aligned} \tag{3.1.3}$$

A can be obtained from the connection form c via $A(X) = \iota(c(X))$ for $X \in T_xM$. Every connection A has a *curvature* F_A, an $\mathrm{ad}(P)$-valued 2-form on M given by

$$F_A = \mathrm{d}A + \frac{1}{2}\left[A \wedge A\right]. \tag{3.1.4}$$

Additionally, A induces a covariant derivative ∇ on any associated bundle $P \times_G V$ by letting for η a vector field on M and $f : P \to V$ a section of V^P

$$\nabla_\eta f = \tilde{\eta} f.$$

Equivalently, it can be understood as an operator $\mathrm{D}_A : \Omega^0(V^P) \to \Omega^1(V^P)$ given by

$$\mathrm{D}_A f = (\mathrm{d} + A) f$$

that extends in the obvious way to higher differentials.

Pure Yang-Mills theory on Minkowski space $M = \mathbb{M}^{n-1,1}$ is defined from the data of a Lie group G together with a bi-invariant inner product $\langle \cdot, \cdot \rangle$ on its Lie algebra $\mathfrak{g}$. A field of the theory is a principal G-bundle over M together with a connection A and its field strength is the curvature F_A. The collection of all fields forms a category $\mathcal{C}_M(G)$ for which

- an object is a connection on a principal G-bundle $P \to M$ and
- a morphism is an isomorphism of principal bundles that preserves the given connections.

Neglecting topological terms, the Yang-Mills Lagrangian is given by

$$L = -\frac{1}{2} \langle F_A \wedge * F_A \rangle \tag{3.1.5}$$

and the equation of motion is

$$\mathrm{D}_A * F_A = 0 \tag{3.1.6}$$

which is known as the *Yang-Mills equation.* Note that $\mathrm{D}_A F_A$ automatically vanishes by the Bianchi identity.

One can combine these two theories into a gauged non-linear σ-model over Minkowski space $M = \mathbb{M}^{1,n-1}$, a very general bosonic theory in the absence of gravity. The data used to define such a theory are:

G	Lie group with Lie algebra $\mathfrak{g}$,	
$\langle \cdot, \cdot \rangle$	bi-invariant inner product on $\mathfrak{g}$,	(3.1.7)
X	Riemannian manifold with a G-action by isometries,	
$V : X \to \mathbb{R}$	G-invariant potential function with minimum 0.	

One obtains fields

A	connection on a principal G-bundle $P \to M$,	(3.1.8)
ϕ	section of the associated bundle $P \times_G X \to M$	

and a Lagrangian by combining (3.1.1) and (3.1.5) into

$$L = \left(\frac{1}{2} |\mathrm{D}_A \phi|^2 - \frac{1}{2} |F_A|^2 - \phi^* V \right) |\mathrm{d}^n x| \tag{3.1.9}$$

In this case, the moduli space of vacua is given by

$$\mathcal{M}_{\mathrm{vac}} = V^{-1}(0)/G. \tag{3.1.10}$$

3.2 BPS states

A general review of representations of $4d$ super Poincaré algebras can be found in B.3, we want to specialize to the case of $\mathcal{N}=2$. For convenience, let us call the super Lie algebra under consideration $\mathfrak{s}$. Consequently, its even part is a direct sum $\mathfrak{s}_0 = \mathfrak{iso}(3,1) \oplus \mathfrak{g}_R \oplus \mathfrak{z}$ of the Poincaré algebra $\mathfrak{iso}(3,1)$, a compact Lie algebra of R-symmetries $\mathfrak{g}_R$ and the center $\mathfrak{z}$, while the odd part $\mathfrak{s}_1$ is a sum of two copies of the irreducible spinor representation together with an action of $\mathfrak{g}_R \oplus \mathfrak{z}$ that does not interest us at this point.

Recall that a massive representation contains two representations of the Clifford algebra according to (B.3.14), there is hence a decomposition $\mathfrak{s}_1 = \mathfrak{s}_1^+ \oplus \mathfrak{s}_1^-$ of the odd part of the superalgebra. Since their actions anticommute, we focus on $\mathfrak{s}_1^+$ and fix two operators a and b to be its creation operators. After choosing a clifford vacuum $|\lambda\rangle$, i.e. $a^\dagger |\lambda\rangle = 0 = b^\dagger |\lambda\rangle$, there is a representation of the Clifford algebra

$$\rho = (|\lambda\rangle, a|\lambda\rangle, b|\lambda\rangle, ab|\lambda\rangle) \tag{3.2.1}$$

which we want to interpret as a representation of $\mathfrak{s}_0' = \mathfrak{so}(3) \oplus \mathfrak{su}(2)_R$, i.e. ρ should be a 4-dimensional representation of the Lorentz algebra. In many physical examples, this is the even part of the little group of the massive state after gauge fixing. Moreover, we will soon focus our attention on BPS states for which the representation theory of the R-symmetry group follows automatically. In either case, one particular way of realizing ρ as such a representation is by letting

$$\rho := \left(0; \frac{1}{2}\right) \oplus \left(\frac{1}{2}; 0\right) \tag{3.2.2}$$

and in fact it has been shown in [WB92] that any representation of $\mathfrak{s}_0' \oplus \mathfrak{s}_1^+$ is of the form $\rho \otimes \sigma$ with σ an arbitrary representation of $\mathfrak{s}_0'$. Including $\mathfrak{s}_1^-$ yields the general form $\rho \otimes \rho \otimes \sigma$ in the rest frame.

Now we specialize to BPS states, i.e. those states that saturate the BPS bound (B.3.15). Hence one Clifford algebra (say $\mathfrak{s}_1^-$) acts trivially and a general BPS representation takes the form

$$\rho_{\text{BPS}} = \rho \otimes \sigma. \tag{3.2.3}$$

We are mainly interested in two examples, called the *hypermultiplet* and the *vector multiplet*. Since BPS are specified by their mass (that is by the $\mathfrak{so}(3)$ part of $\mathfrak{s}_0'$), we can omit the $\mathfrak{su}(2)_R$ part and simplify our notation. For example,

$$\rho = \left(0; \frac{1}{2}\right) \oplus \left(\frac{1}{2}; 0\right) \rightsquigarrow 2 \cdot [0] \oplus \left[\frac{1}{2}\right].$$

In this sense, we can specify long representations λ_j and short (i.e. BPS) representations σ_j for $j \in \frac{1}{2}\mathbb{N}_0$ via

$$\sigma_j := \rho \otimes [j] = \left(2 \cdot [0] \oplus \left[\frac{1}{2}\right]\right) \otimes [j]; \tag{3.2.4}$$

$$\lambda_j := \rho \otimes \rho \otimes [j] = \left(5 \cdot [0] \oplus 4 \cdot \left[\frac{1}{2}\right] \oplus [1]\right) \otimes [j]. \tag{3.2.5}$$

For example,

$$\sigma_0 = \rho \otimes [0] = \rho = 2 \cdot [0] \oplus \left[\frac{1}{2}\right], \tag{3.2.6}$$

$$\sigma_{\frac{1}{2}} = [0] \oplus 2 \cdot \left[\frac{1}{2}\right] \oplus [1], \tag{3.2.7}$$

$$\lambda_0 = 5 \cdot [0] \oplus 4 \cdot \left[\frac{1}{2}\right] \oplus [1], \tag{3.2.8}$$

from which we deduce that $\lambda_0 = 2 \cdot \sigma_0 \oplus \sigma_{1/2}$. σ_0 and $\sigma_{1/2}$ are called *hypermultiplet* and *vector multiplet*, respectively, and it becomes apparent that BPS representations can *fuse* to form non-BPS representations, which however appear "BPS-like" as sums of BPS representations. Our main interest is in the vector multiplet and its content can be read off the above representation: it contains a complex scalar (the [0]), a spinor doublet of $\mathfrak{su}(2)_R$ (the two [1/2]) and a vector (the [1]). As opposed to that, the hypermultiplet contains a doublet of complex scalars and a spinor singlet.

One would like to have an "index" that takes a certain integer value on BPS states and vanishes on non-BPS states and in fact there is a unique (up to a scalar multiple) such index, called the *BPS-index* or *second helicity supertrace*, which was introduced in [CFIV92]. We recall its definition in (3.3.4) but state for now that it receives a contribution of $+1$ for each massive hypermultiplet and -2 for each massive vector multiplet. Consequently (due to linearity), it vanishes for the "fake-BPS" representation λ_0 as expected.

3.3 The BPS index and the Wall-Crossing Formula

We are interested in the low energy behaviour of the theory with gauge group G of rank r (that is low energy compared to a typical energy scale Λ).
The vacuum moduli space is parametrized by the vacuum expectation values (vevs) of the scalar fields. By standard nonrenormalization theorems it is locally a product $\mathcal{M}_H \times \mathcal{B}$. The first factor is called the *Higgs branch* and it is parametrized by the hypermultiplet scalar vevs. In this thesis, we will focus on the second factor, called the *Coulomb branch* and parametrized by the vevs of the vector multiplets. Note that at a generic point $u \in \mathcal{B}$ the Higgs branch is absent. $\mathcal{B}$ is an r-dimensional complex manifold that we will later identify with the base of the Hitchin system for a specific class of theories.

At low energies, the gauge group is broken at each point $u \in \mathcal{B}$ to a maximal torus $\mathrm{U}(1)^r$. According to the Dirac-Zwanziger quantization condition, the corresponding charges span a lattice $\Gamma_u \cong \mathbb{Z}^{2r}$ that comes equipped with a non-degenerate symplectic form $\langle \cdot, \cdot \rangle \to \mathbb{Z}$. These lattices glue to a local system Γ,[1] that is a fibre bundle $\Gamma \to \mathcal{B}$ with fibre Γ_u. BPS particles become massless on singular loci $\mathcal{B}^{\text{sing}}$ of complex codimension one in $\mathcal{B}$. The local system degenerates at these loci and has non-trivial monodromy around them. Let us informally denote local sections of Γ by $\gamma \in \Gamma$. The periods form a collection $Z(u) \in \Gamma_u^* \otimes_{\mathbb{Z}} \mathbb{C}$ holomorphic in $u \in \mathcal{B}$ that allows the definition of the central charge

$$Z_\gamma(u) := Z(u) \cdot \gamma \tag{3.3.1}$$

[1]That is in the absence of flavour charges. More precisely, the full charge lattice Γ has a radical Γ_{f} of flavor charges with respect to the symplectic pairing, and the lattice Γ_g of electric and magnetic gauge charges makes the following a short exact sequence: $0 \to \Gamma_{\mathrm{f}} \to \Gamma \to \Gamma_g \to 0$. Locally, this sequence splits and we will neglect flavor charges until they reappear in (4.4.6).

of a particle with charge $\gamma \in \Gamma$. Any 1-particle state must satisfy the BPS-bound (B.3.15)

$$m \geq |Z_\gamma| \tag{3.3.2}$$

and the particles saturating this bound are exactly the BPS particles. Recall that the one-particle Hilbert space for any state is graded by the charge lattice

$$\mathcal{H}^1_u = \bigoplus_{\gamma\in\Gamma} \mathcal{H}^1_{u,\gamma}. \tag{3.3.3}$$

Since a BPS state may take an arbitrary charge, the space of BPS particles inherits this grading (H is the Hamiltonian of the system):

$$\begin{aligned}\mathcal{H}^{1,\text{BPS}}_u &= \bigoplus_{\gamma\in\Gamma} \mathcal{H}^{1,\text{BPS}}_{u,\gamma},\\ \mathcal{H}^{1,\text{BPS}}_{u,\gamma} &= \left\{ |\phi\rangle \in \mathcal{H}^1_{u,\gamma} : H\,|\phi\rangle = |Z_\gamma(u)|\,|\phi\rangle \right\}.\end{aligned}$$

The $\mathcal{H}^{1,\text{BPS}}_{u,\gamma}$ are finite-dimensional in all known examples and one hence defines the *second helicity supertrace* or *BPS index* by

$$\Omega(u,\gamma) := -\text{tr}_{\mathcal{H}^{1,\text{BPS}}_{u,\gamma}} (-1)^{2J}(2J)^2. \tag{3.3.4}$$

Here, J is the spin operator, i.e. in the notation of (3.2.4)-(3.2.8) it acts on a state $[j]$ as $J[j]=j$. One can easily check that indeed

$$\begin{aligned} -\text{tr}_{\sigma_0}(-1)^{2J}(2J)^2 &= -(-1)^1\cdot 1^2 = 1\\ -\text{tr}_{\sigma_{\frac{1}{2}}}(-1)^{2J}(2J)^2 &= -2\cdot(-1)^1\cdot 1^2 - (-1)^2\cdot 2^2 = -2\end{aligned} \tag{3.3.5}$$

as claimed. The BPS index thus "counts" the BPS states and is hence locally constant. However, there is a codimension 1 locus in $\mathcal{B}$ where the central charges of several BPS states may align (i.e. they have the same phase in the complex plane), causing the binding energy

$$E = |Z_{\gamma+\delta}(u)| - |Z_\gamma(u)| - |Z_\delta(u)| \leq 0 \tag{3.3.6}$$

of a bound BPS state, which is non-positive by the triangle inequality, to vanish. The BPS index is piecewise constant with jumps along these walls, called *walls of marginal stability*. In order to understand the theory well, one would like to understand the global behaviour of $\Omega(u,\gamma)$ but that includes determining its value in strong coupling regions where the methods of computation are often insufficient. Hence, determining the locations of the walls of marginal stability and the jumps of the index is a promising approach to determine the whole spectrum because one would only need to compute the BPS index in one region that is suitable. We now want to describe this jumping behaviour, which was pioneered by the relation to generalized Donaldson-Thomas invariants found by Kontsevich and Soibelman ([KS08]). Let us explain this along the line of [GMN10].
The starting point is a Lie algebra with generators $(e_\gamma)_{\gamma\in\Gamma}$ and commutation relation

$$[e_\gamma, e_\delta] := (-1)^{\langle\gamma,\delta\rangle}\langle\gamma,\delta\rangle e_{\gamma+\delta}. \tag{3.3.7}$$

We want to interpret this Lie algebra geometrically, namely as infinitesimal symplectomorphisms of the complexified torus $T_u = \Gamma^*_u \otimes_{\mathbb{Z}} \mathbb{C}$. We choose coordinates $X^i = X_{\gamma_i}$

with $(\gamma_1,\ldots,\gamma_{2r})$ a basis of the lattice and the X_γ functions on T_u that satisfy

$$X_\gamma X_\delta = X_{\gamma+\delta}. \tag{3.3.8}$$

These allow for a symplectic closed 2-form ω_{T_u} on T_u via

$$\omega_{T_u} := \frac{1}{2}\langle\gamma_i,\gamma_j\rangle^{-1}\mathrm{d}\,\log X^i \wedge \mathrm{d}\,\log X^j \tag{3.3.9}$$

Denoting by $\tilde{e}_\gamma$ the infinitesimal symplectomorphism of the torus generated by X_γ, they naturally act via

$$\tilde{e}_\gamma X_\delta = \langle\gamma,\delta\rangle X_{\gamma+\delta} \tag{3.3.10}$$

and hence satisfy the commutation relations

$$[\tilde{e}_\gamma,\tilde{e}_\delta] = \langle\gamma,\delta\rangle\tilde{e}_{\gamma+\delta}. \tag{3.3.11}$$

The missing sign compared to (3.3.7) has to be absorbed by a quadratic refinement, that is a map $\sigma:\Gamma\to\mathbb{Z}_2$ satisfying

$$\sigma(\gamma)\sigma(\delta) = (-1)^{\langle\gamma,\delta\rangle}\sigma(\gamma+\delta). \tag{3.3.12}$$

Note that this fibration generally does not exist globally, hence one really needs to consider a twisted torus fibration. We will not dwell on this issue which is resolved in [GMN10], p.14. In any case, having chosen a refinement σ allows us to think of e_γ as the symplectomorphism generated by the Hamiltonian $\sigma(\gamma)X_\gamma$. Through the exponential map we can define a group element

$$\mathcal{K}_\gamma := \exp\sum_{n\in\mathbb{N}}\frac{1}{n^2}e_{n\gamma} \tag{3.3.13}$$

which acts as a symplectomorphism on T_u via

$$\mathcal{K}_\gamma\left(X_\delta\right) = X_\delta\big(1-\sigma(\gamma)X_\gamma\big)^{\langle\delta,\gamma\rangle}. \tag{3.3.14}$$

The only ingredients left to describe the wall-crossing formula (WCF) are the walls themselves. As we have seen around (3.3.6), we are interested in the phase of the central charge of a BPS state. Denote therefore the ray in the complex plane that is defined by this phase by

$$l_{\gamma,u} := \left\{\zeta : Z_\gamma(u)\in\zeta\mathbb{R}_-\right\}. \tag{3.3.15}$$

Varying $u\in\mathcal{B}$ rotates these rays in the ζ-plane but keeps their relative ordering invariant around generic u. However, upon reaching a wall of marginal stability two or more central charges become aligned and so do their corresponding rays, switching their order when passing the wall. At a generic such point of the walls u_0, only two central charges align and hence the charges of interest are given by

$$Z_\gamma(u_0) = mZ_{\gamma_1}(u_0) + nZ_{\gamma_2}(u_0)\ ,\quad m,n\in\mathbb{N}_{>0} \tag{3.3.16}$$

for primitive vectors γ_1, γ_2 with aligned central charges $Z_{\gamma_1} \in Z_{\gamma_2}\mathbb{R}_{>0}$.[2] Now the statement of the WCF is the following

Theorem 3.3.1. *(Wall-crossing formula)*
The product

$$A := \prod_{\substack{\gamma = m\gamma_1 + n\gamma_2 \\ m,n>0}}^{\curvearrowright} \mathcal{K}_\gamma^{\Omega(u,\gamma)} , \tag{3.3.17}$$

with the order taken by decreasing argument of the rays l_γ is well-defined and its value is unchanged across the wall.

Recall that at the wall, the BPS indices $\Omega(u, \gamma)$ jump and the order of the product (3.3.17) is changed, so it is anything but obvious that A should remain unchanged. In fact, the statement is strong enough to determine the $\Omega(u, \gamma)$ on one side of the wall given their values on the other side. To see this, one truncates the infinite product to a finite product with $m+n \leq R$, that is we take the quotient of the Lie algebra generated by all the $e_{m\gamma_1+n\gamma_2}$ by the ideal generated by those with $m+n > R$. Similarly, one can compute the value of the power expansion

$$A : X_\delta \to \left(1 + \sum_{0<m,n} c_\delta^{m,n} X_{m\gamma_1+n\gamma_2}\right) X_\delta \tag{3.3.18}$$

on one side of the wall. By a truncation $m + n \leq R$ one can iteratively determine the BPS degeneracies on the other side. For example, for $R = 1$ the values of $c_\delta^{1,0}$ and $c_\delta^{0,1}$ must be correctly reproduced by $\Omega(u, \gamma_1)$ and $\Omega(u, \gamma_2)$, hence fixing them. In the next step $R = 2$ this information can be used to compute $\Omega(u, 2\gamma_1)$, $\Omega(u, \gamma_1 + \gamma_2)$ and $\Omega(u, 2\gamma_2)$ and so on. It is conjectured in [KS08] that the $\Omega(u, \gamma)$ thus computed are actually integers. Let us finish this section with two examples to illustrate why the above claim should hold.

Example 3.3.2. Denote the charges of the two-dimensional lattice that generate the symplectomorphisms at a generic point on the wall of marginal stability by $\gamma = (p, q) \in \mathbb{Z}$ and let $x = X_{(1,0)}$ and $y = X_{(0,1)}$ the functions of unit electric resp. magnetic charge. By mulitplicativity of the X_γ it suffices to study how x and y transform. By (3.3.14) this happens as

$$\mathcal{K}_{(p,q)} : (x, y) \mapsto \Big((1 - (-1)^{pq} x^p y^q)^q \, x, (1 - (-1)^{pq} x^p y^q)^{-p} \, y \Big) \tag{3.3.19}$$

where we have used the canonical symplectic form $\langle (p, q), (p', q') \rangle = pq' - qp'$. Now consider the setting where there are two BPS particles, one with unit electric charge and one with unit magnetic charge, on one side of the wall at which their central charges align. The wall-crossing is described by the "pentagon identity" that has already been observed by Kontsevich and Soibelman:

$$\mathcal{K}_{(1,0)}\mathcal{K}_{(0,1)} = \mathcal{K}_{(0,1)}\mathcal{K}_{(1,1)}\mathcal{K}_{(1,0)}. \tag{3.3.20}$$

[2] As was pointed out in [GMN10], the relationship to generalized Donaldson-Thomas invariants restricts the existence of these primitive vectors by the "Support Property": after choosing a (positive definite) norm on Γ, one needs to require the existence of a positive K such that $Z_\gamma > K \cdot ||\gamma||$ for all γ with $\Omega(u, \gamma) > 0$.

The calculation is straightforward but let us quickly do it:

$$\begin{aligned}\mathcal{K}_{(1,0)}\mathcal{K}_{(0,1)}(x,y) &= \mathcal{K}_{(1,0)}\left((1-y)x,y\right) \\ &= \Big((1-y)x,(1-(1-y)x)^{-1}\,y\Big) \\ &= \Big((1-y)x,(1-x+xy)^{-1}\,y\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{K}_{(0,1)}\mathcal{K}_{(1,1)}\mathcal{K}_{(1,0)}(x,y) &= \mathcal{K}_{(0,1)}\mathcal{K}_{(1,1)}\Big(x,(1-x)^{-1}y\Big) \\ &= \mathcal{K}_{(0,1)}\Big(\Big(1+x(1-x)^{-1}y\Big)\,x, \\ &\qquad\qquad \Big(1+x(1-x)^{-1}y\Big)^{-1}(1-x)^{-1}y\Big) \\ &= \mathcal{K}_{(0,1)}\Big((1-x+xy)\,(1-x)^{-1}x,(1-x+xy)^{-1}\,y\Big) \\ &= \Big(\Big(1-(1-x+xy)^{-1}\,y\Big)\,(1-x+xy)\,(1-x)^{-1}x, \\ &\qquad\qquad (1-x+xy)^{-1}\,y\Big) \\ &= \Big((1-x-y+xy)(1-x)^{-1}x,(1-x+xy)^{-1}\,y\Big) \\ &= \Big((1-y)x,(1-x+xy)^{-1}\,y\Big).\end{aligned}$$

Hence, the creation of a dyon (a particle carrying both electric and magnetic charges) is predicted by the WCF which is in accordance with supergravity considerations [DM11] that also find this bound state.
A second remarkable example arises in Seiberg-Witten theory which contains one wall of marginal stability separating the weak coupling and the strong coupling regime. The wall-crossing occurring there is expressed by

$$\mathcal{K}_{(2,-1)}\mathcal{K}_{(0,1)} = \Big(\mathcal{K}_{(0,1)}\mathcal{K}_{(2,1)}\mathcal{K}_{(4,1)}\dots\Big)\,\mathcal{K}_{(2,0)}^{-2}\,\Big(\dots\mathcal{K}_{(6,-1)}\mathcal{K}_{(4,-1)}\mathcal{K}_{(2,-1)}\Big). \tag{3.3.21}$$

The left hand side represents the strong coupling spectrum and contains one monopole and one dyonic state, both in a hypermultiplet representation (because of the exponent). The weak coupling spectrum is represented by the right hand side with the infinite spectrum of dyons (all in the hypermultiple representation) and a state with charge $(2,0)$ in the vectormultiplet representation, hence the W boson (since the scalars are neutral in the vectormultiplet).

3.4 Seiberg-Witten theory

In this section we are interested in the bosonic degrees of freedom at low energies, that is in the complex scalar fields a^I and gauge fields A^I, all taken in the adjoint representation. As before, we make the genericity assumption that the scalars have non-vanishing vacuum expectation values, hence breaking the gauge group to a maximal torus $U(1)^r$. Seiberg-Witten theory ([SW94a], [SW94b]) explains that the effective Lagrangian is encoded in a single holomorphic function $\mathcal{F}(a^I)$, called the *prepotential*. Let us quickly recall the main points.

We have seen in the previous section that the Coulomb branch $\mathcal{B}$ comes equipped with a fibre bundle of lattices $\Gamma_u = \mathbb{Z}^{2r}$ which carries a non-degenerate symplectic form.

Locally we can hence choose a splitting into dual Lagrangian sublattices $\Gamma = \Gamma_e \oplus \Gamma_m$ called the lattices of *electric* resp. *magnetic charges*. We can choose a Darboux basis $\{\alpha_I, \beta^I\}$ (called an electric-magnetic duality frame in physics) which obeys the following relations under the symplectic pairing:

$$\langle \alpha_I, \alpha_J \rangle = 0 = \langle \beta^I, \beta^J \rangle \,, \quad \langle \alpha_I, \beta^J \rangle = \delta_I^J. \tag{3.4.1}$$

In these coordinates the central charge Z can be written as

$$Z = a^I \alpha_I - b_I \beta^I \tag{3.4.2}$$

and we can reinterpret $Z_\gamma = \langle Z, \gamma \rangle$. The central charge section hence determines special local coordinates on $\mathcal{B}$ given by

$$a^I = Z_{\beta^I}. \tag{3.4.3}$$

By Seiberg-Witten theory there exists a holomorphic function $\mathcal{F}(a^I)$ such that

$$b_I = Z_{\alpha_I} = \frac{\partial \mathcal{F}}{\partial a^I}. \tag{3.4.4}$$

This determines a restriction on Z of the form

$$\begin{aligned} \langle \mathrm{d}Z, \mathrm{d}Z \rangle &= -2\mathrm{d}a^I \wedge \mathrm{d}b_I = -2\frac{\partial b_I}{\partial a^J}\mathrm{d}a^I \wedge \mathrm{d}a^J \\ &= -2\frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J}\mathrm{d}a^I \wedge \mathrm{d}a^J = 0 \end{aligned}$$

which means that $\mathcal{B}$ is a *special Kähler manifold* with Kähler form

$$\omega = \frac{i}{2\pi}\partial\bar{\partial}\langle Z, \bar{Z}\rangle. \tag{3.4.5}$$

This was first realized in [Str90], see also [Fre99]. The prepotential determines the symmetric gauge coupling matrix

$$\tau_{IJ} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J} \tag{3.4.6}$$

which allows to write the bosonic part of the 4-dimensional effective Lagrangian as

$$\mathcal{L}^{(4)} = -\frac{\mathrm{Im}\tau_{IJ}}{4\pi}\left(\mathrm{d}a^I \wedge *\mathrm{d}\bar{a}^J + F^I \wedge *F^J\right) + \frac{\mathrm{Re}\tau_{IJ}}{4\pi}F^I \wedge F^J \tag{3.4.7}$$

with the field strength $F^I := \mathrm{d}A^I$.

3.5 Dimensional reduction to three dimensions

We now want to compactify this theory on a circle S^1_R of radius R to obtain an effective three-dimensional theory at energies much smaller than any other scales, i.e. at energies $E \ll 1/R$ and $E \ll \Lambda$ (the energy scale of the four-dimensional theory). This process has not yet been carried out on a mathematical level of rigour, a good reference for the state of the art is [Nak15].

Call the Coulomb branch of the three-dimensional theory $\mathcal{M}$. We obtain a first estimate by naive dimensional reduction, i.e. by imposing that all the fields in (3.4.7) split into a part that depends on the three remaining directions and a part that depends solely on the fourth (compactified) direction. Let us choose a local coordinate frame $\{x^0, .., x^3\}$ with the x^3-direction being compactified. We will impose the following (where from now on d denotes the exterior derivative in three dimensions, d_4 that in four dimensions):

$$\begin{aligned} \mathrm{d}_4 a^I &\longrightarrow \mathrm{d}a^I_{(3)}, \\ A^I = A^I_\mu \mathrm{d}x^\mu &\longrightarrow A^I_{(3)} + A^I_4 \mathrm{d}x^4, \\ F^I = \mathrm{d}_4 A^I &\longrightarrow F^I_{(3)} + \mathrm{d}A^I_4 \wedge \mathrm{d}x^4. \end{aligned}$$

Note that in particular the gauge fields A^I split into 1-forms $A^I_{(3)}$ and scalars A^I_4. This is important because the moduli space is made up of scalar fields and the Coulomb branch consists of vector multiplets which contain the same amount of scalars as of gauge fields. Hence, when passing to the three-dimensional theory, the new scalars A^I_4 will cause the dimension of the three-dimensional Coulomb branch to double compared to the four-dimensional one. More precisely, the new scalar degrees of freedom are periodic and are given by "electric" and "magnetic" Wilson lines (holonomies)

$$\theta^I_e := \int_{S^1} A^I_4 \mathrm{d}x^4, \quad \theta_{m,I} := \int_{S^1} \left(A^*_{(3)}\right)_I \mathrm{d}x^4, \tag{3.5.1}$$

where the $\left(A^*_{(3)}\right)_I$ are scalars dual to the 1-forms $A^I_{(3)}$. We will not make this duality precise here but note that in particular

$$\mathrm{d}\theta_{m,I} = \mathrm{Re}\tau_{IJ}\mathrm{d}\theta^J_e + * \left(\mathrm{Im}\tau_{IJ}F^J\right). \tag{3.5.2}$$

In any case, for generic $u \in \mathcal{B}$ one obtains a $2r$-torus $\mathcal{M}_u$ parametrized by these periodic scalars. Varying u one obtains a fibration $\mathcal{M} \to \mathcal{B}$ with generic fiber $\mathcal{M}_u$ a $2r$-torus. However, at singular points the fiber might degenerate. To make this more precise, introduce $\mathrm{d}z_I = \mathrm{d}\theta_{m,I} - \tau_{IJ}\mathrm{d}\theta^J_e$ with a slight abuse of notation since these 1-forms are not closed globally but only along fibers $\mathcal{M}_u$. Moreover, to ease the notation we will drop indices and adopt the notation $\tau|\mathrm{d}a|^2 := \tau_{IJ}\mathrm{d}a^I \wedge *\mathrm{d}a^J$. By integrating out the S^1-direction and some rearranging (see [Til13] for a derivation) we obtain the effective three-dimensional Lagrangian

$$\mathcal{L}^{(3)} = -\frac{1}{2}\left\{R\,(\mathrm{Im}\tau)\,|\mathrm{d}a|^2 + \frac{1}{4\pi^2 R}\,(\mathrm{Im}\tau)^{-1}\,|\mathrm{d}z|^2\right\}. \tag{3.5.3}$$

Recalling definition 3.1.1, we can identify this theory as a σ-model with target space $\mathcal{M}$ and fields a^I and z_I which are maps $\mathbb{R}^{2,1} \to \mathcal{M}$. The metric (now as a 2-form) must then locally take the form

$$g^{\mathrm{sf}} = R\,(\mathrm{Im}\tau)\,|\mathrm{d}a|^2 + \frac{1}{4\pi^2 R}\,(\mathrm{Im}\tau)^{-1}\,|\mathrm{d}z|^2. \tag{3.5.4}$$

This metric gives away two interesting properties of the moduli space $\mathcal{M}$: Firstly, the second summand tells that the torus fibers are flat, hence the metric is called *semiflat*. Secondly, g^{sf} is a Kähler metric with respect to a complex structure in which $\mathrm{d}a^I$ and $\mathrm{d}z_I$ are a basis for the holomorphic one forms $\Omega^{1,0}$. Moreover, the tori $\mathcal{M}_u$ are complex submanifolds with respect to this complex structure. The description of g^{sf} is local but they glue together to a globally smooth metric on $\mathcal{M}$ except over the singular loci of

$\mathcal{B}$ where g^{sf} has a singularity. This raises a puzzle because $\mathcal{M}$ should have a globally well-defined metric g. Indeed, one needs to include "BPS instanton" corrections, i.e. consider cases where the world lines of BPS particles wrap around the S^1 over which the compactification was worked out.

Let us describe how to include the fermionic degrees of freedom, i.e. supersymmetry. There are 8 real supersymmetry charges in the four-dimensional $\mathcal{N} = 2$ theory in the Weyl spinors Q^I_α and $\bar{Q}^{J\dot{\beta}}$ because the indices take values in $\{1, 2\}$. All these supercharges are preserved upon compactification on S^1 but are now organized in the three-dimensional spinors. The irreducible spin representation is a Majorana spinor with 2 supercharges, hence one is led to four spinors, i.e. $\mathcal{N} = 4$ supersymmetry [SW97].

There is one last thing we have not talked about yet, namely the R-symmetry group of the theory. Seiberg and Witten ([SW97]) explain how $\mathcal{M}$ obtains a Hyperkähler structure in terms of it by considering the four-dimensional theory itself to be derived via a dimensional reduction. More precisely, when starting from a six-dimensional super Yang-Mills theory with $\mathcal{N} = 1$ supersymmetry, we obtain the four-dimensional theory by imposing the fields to be independent of two coordinates, say x^5 and x^4. This gives rise to an R-symmetry group $\mathrm{U}(1)_R$ that encodes rotations in the $x^4 - x^5$-plane. However, upon additional dimensional reduction, i.e. imposing independence of the x^3 coordinates as well, one obtains an $\mathrm{SO}(3)$ that acts via rotations on the last three coordinates. Hence, the $\mathrm{SO}(3)$ must act on the scalars in the three-dimensional theory and thus on the geometry of $\mathcal{M}$ by rotating the complex structures. This way, there is an S^2 worth of complex structures as would be expected for a Hyperkähler manifold. This is a very typical feature of supersymmetry and there is in fact an entire book about the interplay between supersymmetry and the underlying geometry [Cor10]. As an example, in four-dimensional σ-models with $\mathcal{N} = 1$ supersymmetry the scalars parametrize a Kähler manifold, while for $\mathcal{N} = 2$ it needs to be Hyperkähler.

3.6 Construction of the Hyperkähler metric

To fully describe the Hyperkähler structure of $\mathcal{M}$, one needs to obtain the exact description of its metric g. This is the main content of [GMN10] where a twistorial construction of Hyperkähler metrics is used. Recall that the twistor space $\mathcal{T}$ assigned to a Hyperkähler manifold $\mathcal{M}$ is topologically the product $\mathcal{M} \times \mathbb{CP}^1$. From the point of view of complex analysis one must first identify the complex structures $J^{(\zeta)}$ that make $\mathcal{M}$ a Kähler manifold with $\mathbb{CP}^1$. Then the twistor space should be understood as a holomorphic fibration over $\mathbb{CP}^1$ with the fiber being a copy of $\mathcal{M}$ endowed with the complex structure $J^{(\zeta)}$ corresponding to the base point $\zeta \in \mathbb{CP}^1$. The holomorphic sections of this fibration are called *twistor lines*.

The approach by Gaiotto, Moore and Neitzke [GMN10] takes the reverse direction: Starting from the space $\mathcal{M} \times \mathbb{CP}^1$ and given functions

$$\mathcal{X}_\gamma : \mathcal{M} \times \mathbb{C}^\times \to \mathbb{C}^\times$$

labeled by sections $\gamma \in \Gamma$ that satisfy certain conditions (one of them being that they vary holomorphically with the twistor parameter $\zeta \in \mathbb{C}^\times$, i.e. they make use of the complex structure of $\mathbb{C}^\times \subset \mathbb{CP}^1$), it is possible to identify $\mathcal{M} \times \mathbb{CP}^1$ with the twistor space $\mathcal{T}$ assigned to the manifold $\mathcal{M}$ which is hence Hyperkähler. This uses a characterization by Hitchin et al ([HKLR87], [Hit92]). Moreover, it is possible to extract the Hyperkähler metric g from this construction that will make the $\mathcal{X}_\gamma(\cdot; \zeta)$ holomorphic Darboux coordinates[3] in the complex structure $J^{(\zeta)}$ and that makes the following a holomorphic symplectic form in that complex structure:

$$\varpi(\zeta) := \frac{1}{8\pi R^2}\epsilon_{ij}\frac{\mathrm{d}\mathcal{X}_{\gamma^i}}{\mathcal{X}_{\gamma^i}} \wedge \frac{\mathrm{d}\mathcal{X}_{\gamma^j}}{\mathcal{X}_{\gamma^j}}. \tag{3.6.1}$$

Here, d denotes the fibrewise differential at fixed value of ζ. To make the above statement precise, fix an open subset $U \in \mathcal{B}$ over which the charge lattice fibration Γ is trivializable. Recall that $\mathcal{M}$ is a fibration $\pi : \mathcal{M} \to \mathcal{B}$ with generic fibre a $2r$-torus. We denote local coordinates on $\pi^{-1}(U)$ by (u, θ) with $u \in U$ and θ the angular coordinates in the torus fibre $\mathcal{M}_u$. [GMN10] demand the existence of functions

$$\mathcal{X} : \mathcal{M} \times \mathbb{C}^\times \to \Gamma^* \times \mathbb{C}^\times$$

that are holomorphic in $\zeta \in \mathbb{C}^\times$ and satisfy the following properties (where $\mathcal{X}_\gamma$ denotes the contraction with $\gamma \in \Gamma$):

(A) They are linear in γ, i.e.

$$\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'}; \tag{3.6.2}$$

(B)

$$\mathcal{X}_\gamma(\cdot;\zeta) = \overline{\mathcal{X}_{-\gamma}\left(\cdot;\overline{-1/\zeta}\right)}; \tag{3.6.3}$$

(C) All $\mathcal{X}$ simultaneously satisfy the following set of differential equations:

$$\begin{aligned} \frac{\partial}{\partial u^i}\mathcal{X} &= \left(\frac{1}{\zeta}A^{(-1)}_{u^i} + A^{(0)}_{u^i}\right)\mathcal{X}, \\ \frac{\partial}{\partial \bar{u}^{\bar{i}}}\mathcal{X} &= \left(A^{(0)}_{\bar{u}^{\bar{i}}} + \zeta A^{(1)}_{\bar{u}^{\bar{i}}}\right)\mathcal{X}, \end{aligned} \tag{3.6.4}$$

with the $A^{(n)}_{u^i}$ and $A^{(n)}_{\bar{u}^{\bar{i}}}$ being certain complex vertical vector fields on the torus fibres $\mathcal{M}_u$. The motivation behind (3.6.4) is that they can be reinterpreted as the Cauchy-Riemann equations on $(\mathcal{M}, g)$ in the complex structure $J^{(\zeta)}$.

(D) For fixed (u, θ), the assignment $\zeta \mapsto \mathcal{X}_\gamma(u, \theta; \zeta)$ is holomorphic on a dense subset of $\mathbb{C}^\times$ (actually away from a countable union of lines);

(E) $\varpi(\zeta)$ glues to a globally defined function that is holomorphic in ζ;

(F) $\ker\varpi(\zeta)$ is a $2r$-dimensional subspace of the $4r$-dimensional $T_{\mathbb{C}}\mathcal{M}$, as would be expected for a non-degenerate holomorphic symplectic form;

(G) $\varpi(\zeta)$ has only simple poles in the limits $\zeta \to 0$ and $\zeta \to \infty$.

The statement is that given such a collection of functions $\mathcal{X}_\gamma$, $\mathcal{M}$ is Hyperkähler with a metric g such that for fixed twistor parameter ζ, $\mathcal{X}_\gamma(\cdot;\zeta)$ become holomorphic Darboux coordinates and $\varpi(\zeta)$ becomes a holomorphic symplectic form in the complex structure $J^{(\zeta)}$. Hence the problem of constructing the metric g has been transformed into that of finding the coordinate functions $\mathcal{X}$.

[3] We will stick to the name Darboux coordinates that has been used by Gaiotto, Moore and Neitzke. Note however that the coordinates on the holomorphic symplectic manifold $\mathcal{M}$ guaranteed by Darboux's theorem should really be $\log \mathcal{X}_\gamma$.

A first estimate is given by the *semiflat* Darboux coordinates that are defined locally as

$$\mathcal{X}_\gamma^{\text{sf}}(u,\theta;\zeta) = \exp\left(\pi R\zeta^{-1}Z_\gamma(u) + i\theta_\gamma + \pi R\zeta\bar{Z}_\gamma(u)\right) \tag{3.6.5}$$

and which correctly determine a Hyperkähler metric on $\mathcal{M}$, namely the semiflat metric g^{sf} that we have encountered in (3.5.4).

As we have stated before, in order to determine the quantum corrected metric g (in which the real codimension two singularities of g^{sf} are smoothed out) one needs to include instanton corrections. This cannot be recalled in a precise way in the scope of this thesis but we want to emphasize some main features. We will denote the holomorphic Darboux coordinates that are featured in [GMN10] by $\mathcal{X}_\gamma^{\text{RH}}(u,\theta;\zeta)$ in accordance with [GMN13c]. The most important point is that $\mathcal{X}_\gamma^{\text{RH}}(u,\theta;\zeta)$ are not holomorphic in ζ on all of $\mathbb{C}^\times$ but only piecewise holomorphic with discontinuities at *BPS-rays*

$$l_{\gamma_{\text{BPS}},u} := \left\{\zeta \in \mathbb{C}^\times : \zeta \in Z_{\gamma_{\text{BPS}}}(u)\mathbb{R}_-\right\}. \tag{3.6.6}$$

where the Darboux coordinates jump. The crucial point is that continuity of the metric is equivalent to these jumps obeying the KSWCF (3.3.17). More precisely, define the *KS transformations*

$$\mathcal{K}_\gamma : \mathcal{X}_{\gamma'}^{\text{RH}} \to \mathcal{X}_{\gamma'}^{\text{RH}}\left(1 - \sigma(\gamma)\mathcal{X}_\gamma^{\text{RH}}\right)^{\langle\gamma',\gamma\rangle}. \tag{3.6.7}$$

Then the Darboux coordinates jump by

$$\mathbf{S}_{\gamma_{\text{BPS}},u} := \prod_{\gamma||\gamma_{\text{BPS}}} \mathcal{K}_\gamma^{\Omega(u,\gamma)} \tag{3.6.8}$$

at the BPS ray $l_{\gamma_{\text{BPS}},u}$. Note that we do not need to fix an ordering here because all KS transformations commute due to $\langle\gamma,\gamma'\rangle = 0$ for aligned charges $\gamma||\gamma'$. Moreover, we demand that $\mathcal{X}_\gamma^{\text{RH}} \sim \mathcal{X}_\gamma^{\text{sf}}$ in the limits $\zeta \to 0,\infty$ as well as in the limit $R \to \infty$.

The upshot is that we are looking for a collection of functions $\mathcal{X}_{\gamma,i}$ that are holomorphic in a sector S_i of the complex plane and that obey boundary conditions at the walls between the sectors as $\mathcal{X}_{\gamma,i+1} = \mathbf{S}_i\mathcal{X}_{\gamma,i}$ and have a certain behaviour in the limits $\zeta \to 0,\infty$. We have hence formulated a Riemann-Hilbert problem, which also explains the name $\mathcal{X}^{\text{RH}}$. It is shown in [GMN10] that this Riemann-Hilbert problem is equivalent to an integral equation that is a special version of the Thermodynamic Bethe Ansatz (TBA). There is a solution in form of a series expansion provided that R is large enough. Hence [GMN10] provide the holomorphic Darboux coordinates that in turn determine the Hyperkähler metric g for large R. In the next section we will turn our attention to a special class of four-dimensional $\mathcal{N}=2$ theories called *Theories of class $\mathcal{S}$* for which a different approach leads to a Hyperkähler metric for arbitrary values of R. The main mathematical ingredient is that the moduli space $\mathcal{M}$ can be identified with the moduli space of solutions to Hitchin's self-duality equations.

Chapter 4

Theories of class $\mathcal{S}$

This chapter is dedicated to the *Theories of Class* $\mathcal{S}$, a certain class of four-dimensional quantum field theories with $\mathcal{N} = 2$ supersymmetry that has first been described by Gaiotto, Moore and Neitzke [GMN13c]. The characteristic feature of these theories is that the fiber bundle $\mathcal{M} \to \mathcal{B}$ can be identified with the Hitchin integrable system which gives rise to very nice geometry. To see this, one must first understand that every theory of class $\mathcal{S}$ arises as the compactification (and some partial topological twisting) of a very mysterious superconformal six-dimensional theory which we call *theory* $\mathfrak{X}$ and introduce in section 4.1. After introducing an enhancement of QFTs by allowing for defects insection 4.2, we will be able to describe how the compactification of the six dimensional theory over a punctured Riemann C surface yields the theory of class $\mathcal{S}$ in section 4.3. The further compactification over a circle defines a three-dimensional σ-model with target space the Hyperkähler manifold $\mathcal{M}$. This manifold can be characterized as the moduli space of solutions of Hitchins equations (4.4.1) by reversing the order of compactification: The compactification of the six dimensional theory over the circle yields a five-dimensional Super Yang-Mills theory and further compactification over C yields the same three-dimensional theory. It is known from physics that $\mathcal{M}$ describes certain BPS configurations of the five-dimensional theory and these equations are simply Hitchins equations. After describing this insight and its first implications in section 4.4, we explain in section 4.5 what the lift of BPS states to the six-dimensional theory is. As it turns out, they are captured by a certain network of curves on the Riemann surface C called a (finite) string web. We will be able to apply this directly in the A_1 case in section 4.6 where the string web is "dual" to a certain decorated triangulation of C and one can hence obtain the Darboux coordinates $\mathcal{X}_\gamma$ from Fock-Goncharov coordinates. In this picture, jumps at the walls where BPS particles sit correspond to certain morphisms of the triangulation which gives another interpretation of the KSWCF.

4.1 The Theory $\mathfrak{X}$

In this section we will describe a class of six-dimensional superconformal theories (SCFTs) $\mathfrak{X}_\mathfrak{g}$ with $\mathcal{N} = (2, 0)$ supersymmetry that are labelled by the choice of a simple, simply laced Lie algebra $\mathfrak{g}$.[1] The classification of superconformal algebras (see table B.2) provides the existence of superconformal algebras of type $\mathcal{N} = (k, 0)$ in six dimensions,

[1] More precisely, it is pointed out in [Moo12] that one should demand:

1. A reductive real Lie algebra $\mathfrak{g} = \mathfrak{z} + \mathfrak{g}'$ with centre $\mathfrak{z}$ and compact semi-simple part $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$;
2. A non-degenerate Ad-invariant bilinear form B normalized to make all roots be of length 2;
3. A certain lattice $\Pi = \Gamma + \Gamma'$ with $\Gamma \subset \mathfrak{z}$ and $\Gamma' \subset \mathfrak{g}'$.

and similar to the four-dimensional case one would need to include gravity for a theory with $k > 2$. Hence, $\mathcal{N} = (2,0)$ is the maximal amount of supersymmetry for a six-dimensional superconformal theory. That being said, it is a different task to find an SCFT that realizes this maximal amount of supersymmetry. Indeed, this has proven to be a very elaborate task and it is an active area of research to define these theories in more detail. We will hence only collect some ideas on this topic in this section and want to emphasize that we will not only be lacking mathematical rigour but also be leaving the ground of what can be considered settled, well-understood physics. The difficulty in understanding these theories is reflected in the name "Theory $\mathfrak{X}$".

The main issue with the theories $\mathfrak{X}_{\mathfrak{g}}$ is that there is no known description in form of an action functional. Hence, any information that is available so far is obtained by indirect means that include the study of dualities, representations of the superconformal algebra and different compactification limits. In fact it is believed that certain compactifications to lower dimensions encode interesting mathematical dualities by changing the order of compactification. Examples include the Geometric Langlands Correspondence [KW07] and the AGT correspondence [AGT10]. We will make use of a construction like that later in this chapter, first we want to explain how to construct these theories starting from *M-theory*, an 11-dimensional hypothetical quantum theory $\mathfrak{M}$ that has the following properties:

1. $\mathfrak{M}$ is not a superconformal theory but enjoys an $\mathfrak{iso}(1,10|32)$ Super-Poincaré symmetry;

2. The characteristic length scale of $\mathfrak{M}$ is the 11-dimensional Planck length ℓ_m;

3. $\mathfrak{M}$ contains two types of branes: M2-branes and M5-branes, i.e. maps $\phi : S \to T$ between a *space-time* manifold S of 3 resp. 6 dimensions and a *target space* manifold T of 11 dimensions. $\phi(S)$ is also called the *world volume*. Generally, these manifolds come endowed with various additional structures such as gauge bundles, tensor fields or a supermanifold extension;

Hence the world volume of a single M5-brane determines a six-dimensional theory and so does more generally a stack of K M5-branes[2]. The physical insight is that this theory contains 16 supercharges and loses the dimensionful parameter (i.e. the characteristic length) and is hence $(2,0)$ superconformal with an R-symmetry group that contains Spin(5) (for the action on the transverse directions). In fact it is believed to be the theory $\mathfrak{X}_{\mathfrak{g}}$ attached to the Lie algebra $\mathfrak{g} = \mathfrak{u}(K)$ which is the case we are interested in in this thesis.

We finish this section by recalling some properties that are crucial for our further understanding. These can be found in more detail and with some explanation to their origins in [Moo12] where they are part of an attempt to axiomatize these theories:

1. Compactification on S^1 yields five-dimensional supersymmetric Yang-Mills theory (SYM) with compact gauge group G_{adj}.

2. For simple $\mathfrak{g}$ the moduli space of vacua on $\mathbb{M}^{1,5}$ takes the form

$$\left(\mathbb{R}^5 \times \mathfrak{t}\right)/\mathcal{W}$$

with $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra and $\mathcal{W}$ the Weyl group.

[2]The notion of "stack" here should not be confused with the mathematical notion of a stack as in an algebraic stack or a differentiable stacks. We merely mean a collection of K parallel M5 branes.

3. The theory carries surface defects $\mathbb{S}_{\vec{n}}(X_2, \mathcal{R})$ which are labelled by
   - the choice of a smooth oriented 2-manifold X_2 smoothly embedded in the 6-manifold X_6 that the theory is evaluated on,[3]
   - a representation $\mathcal{R}$ of $\mathfrak{g}$ and
   - a map $\vec{n} : X_2 \to \mathbb{R}^5$ into the fundamental representation of the Spin(5) R-symmetry group.

   There are certain restrictions on the data that determine (among other things) the amount of supersymmetry preserved.

4. There are also defects of codimension two present, labelled by conjugacy classes of Lie algebra homomorphisms $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}$, monomorphisms $\mathfrak{so}(2) \hookrightarrow \mathfrak{so}(5)$ and a choice of *mass deformation* $\mathfrak{m} \in \mathfrak{t}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$. In particular, these defects preserve a superconformal symmetry isomorphic to $\mathfrak{su}(2, 2|2) \subset \mathfrak{osp}(6, 2|4)$.

Clearly, the last two points demand some clarification on the concept of a defect. We will dwell on this point in the next section. Let us first remark that in order to understand the theories $\mathfrak{X}_{\mathfrak{g}}$ better, one would also need to understand different approaches of construction (see [Moo12] for several recent developments) of which two noteworthy are:

- Via the AdS/CFT correspondence the theories $\mathfrak{X}_{\mathfrak{g}}$ for $\mathfrak{g} = A_K$ or D_K are supposed to be described by their dual description of M-theory on the space $\mathrm{AdS}_7 \times S^4$ resp. $\mathrm{AdS}_7 \times \mathbb{RP}^4$ at least for sufficiently large K. This sheds particular light on the operator spectrum and the superconformal index of the 6d theory. The foundations of this correspondence go back to Maldacena ([Mal98b],[Mal98a]), a more modern and mathematical approach is in [FSS14] where tools such as non-abelian differential cohomology and higher stacks are brought to use.

- Witten describes a construction from the 10-dimensional type IIB string theory [Wit96]. Essentially one is compactifying over a K3 surface, however in the process the K3 manifold develops an ADE singularity which is claimed to be compensated by strong interaction effects, giving rise to a six-dimensional theory of the desired type.

4.2 QFTs with defects

We will describe defects in the setting that emphasizes the extended structure of QFTs as described in 2.2 and neglect the dependence on a geometrical structure. We will merely demand that the manifolds are enhanced with a notion of distance, e.g. by the conformal class of a metric.

Firstly, what does it mean for a smooth manifold or a cobordism to have a defect? A defect is nothing but an embedded submanifold of positive codimension, hence defects themselves can contain defects. Take as an example the manifolds

$$M_k := \{0\}^{n-k} \times \mathbb{R}^k \tag{4.2.1}$$

with the obvious embeddings $M_k \hookrightarrow M_{k+1}$. This way M_i is a defect for M_j for any pair $i < j$. A second example is the boundary ∂M of a manifold with boundary M. More

[3] X_2 is very consequently denoted by Σ in the standard literature but we will reserve Σ for the Seiberg-Witten curve of our four-dimensional theory, that is the spectral curve defined in (4.4.2).

generally, we allow for manifolds with singularities and henceforth call these singularities defects.

More interesting is what it means to have an n-dimensional TQFT with defects. Loosely speaking one absorbs the defects into the definition of the bordism category by allowing for manifolds and bordisms with defects and certain extra data and making the functor dependent on this input. In the example of a codimension one defect, one speaks of a *domain wall*. It allows for two different theories on each side of the wall and the extra data attached to the defect is in this case a prescription on how to "connect" the two theories. In the particular case where the theory on one side is trivial, this reduces to boundary conditions for the non-trivial theory.

A physical insight claims that a k-dimensional defect should be an object in a k-category [Kap10]. Let us repeat the important argument: Consider for simplicity the theory on $\mathbb{R}^n$ with the k-defect given by $D = M_k$. The complement $\mathbb{R}^n \backslash D$ is diffeomorphic to $\mathbb{R}^k \times S^{n-k-1} \times \mathbb{R}_{>0}$ through "polar coordinates". Regularization demands to replace the defect by a boundary condition on the boundary of a suitable tubular neighbourhood $D_\epsilon := \mathbb{R}^k \times S^{n-k-1} \times (0, \epsilon)$ for an $\epsilon > 0$. Compactification over the factor S^{n-k-1} yields a $(k+1)$-dimensional TQFT. More importantly, there is a k-category $\mathcal{C}_\epsilon$ given by evaluation of the bulk TQFT on ∂D_ϵ and the defect is associated with the limit $\epsilon \to 0$ of these k-categories.

Another important example in this spirit is that of 0-defects which are called *local operators* in physics. This gives an interpretation of the *operator-state correspondence* which states that the local operators at a point are in 1:1 correspondence with the elements of the vector space associated to the linking sphere S^{n-1}, i.e. the space of states. Similarly, one-dimensional defects are called line defects and contain such examples as Wilson line operators which are given by the holonomy around a closed curve (see 3.5.1) and two-dimensional defects are called surface defects. Let us finish this section with three remarks:

Remark 4.2.1. As we have stated above, defects can themselves contain defects. As an example, consider an n-dimensional fully extended TQFT Z and the manifolds M_k from (4.2.1). Furthermore, let

$$M_k^\pm := \{0\}^{n-k} \times \mathbb{R}^{k-1} \times \mathbb{R}_\pm \subset M_k$$

be the two components of M_k on either side of the defect M_{k-1}. As we have just explained, M_{n-1} can be considered as an object in an $(n-1)$-theory and so can $M_{n-1}^\pm$. Now M_{n-2} is a codimension one defect inside M_{n-1}, hence an object of an $(n-2)$-category. This is in accordance with two interpretations simultaneously:

- M_{n-2} is an $(n-2)$-defect in the bulk n-dimensional theory and should hence be an object of an $(n-2)$-category by the argumentation above;

- M_{n-2} also provides a morphism between two objects of the same $(n-1)$-category, namely the objects corresponding to $M_{n-1}^\pm$. Since the morphism space of an $(n-1)$-category is per definition an $(n-2)$-category, M_{n-2} should be associated to an object in the latter.

Hence a codimension two defect may be interpreted as a domain wall between domain walls. Going on in this matter, these themselves can carry the defects M_{n-3} which are morphisms in the $(n-2)$-category and so forth.

Remark 4.2.2. The definition of TQFTs with defects changes the corresponding Cobordism Hypothesis dramatically. Recall that theorem 2.2.17 states that the plain (∞, n)-category of cobordisms[4] is a symmetric monoidal (∞, n)-category that is freely generated by a single object, namely the 0-morphism represented by a point. Lurie [Lur09, § 4.3] has proposed a formalization of the notion of defects in the purely topological setting together with an altered version of the Cobordism Hypothesis for this case which states that every included k-defect adds a k-morphism generator to the bordism (∞, n)-category.

Remark 4.2.3. The origin of the defects (or at least a certain subclass of them) in the theories $\mathfrak{X}_\mathfrak{g}$ can be described very visually in the setting where the theory describes the world volume theory of K coincident M5-branes:

- The codimension two defects arise when another M5 brane is added to the configuration that intersects the stack of K M5-branes transversely in codimension two while preserving supersymmetry. Hence the single M5-brane fills a 2-plane in the $\mathbb{R}^5$ transverse to the stack of M5-branes, and the Grassmannian parametrizes the monomorphism $\mathfrak{so}(2) \hookrightarrow \mathfrak{so}(5)$.

- Similarly, surface defects can arise by intersection with M5-branes in two-dimensional subspaces. In a different spirit, the dimension two defects can origin from enhancing the stack of M5-branes by M2-branes, the other type of branes present in M-theory. The idea is that an M2-brane can stretch between two neighbouring M5-branes, ending on a subsurface of each M5-brane and extending in one transverse direction.

4.3 Compactify to $4d$: Theories of class $\mathcal{S}$

In this section we want to introduce a special class of four-dimensional field theories with $\mathcal{N} = 2$ supersymmetry, called the theories of class $\mathcal{S}$. We denote them by $S(\mathfrak{g}, C, D)$ where

- $\mathfrak{g}$ is a simple simply laced Lie algebra, we will restrict our attention to the case $A_{K-1} = \mathfrak{su}(K)$;

- C is a Riemann surface with a finite number of punctures $\mathfrak{s}_1, \ldots, \mathfrak{s}_n$ (we will always assume $n \geq 1$);

- Each puncture $\mathfrak{s}_k$ should be realised as a codimension two defect of the six-dimensional theory $\mathfrak{X}_\mathfrak{g}$, hence we have to add the "data" $D_k = D(\rho_k, [V], \mathfrak{m}^{(k)})$ to it that is labelled by a choice of representation $\rho_k : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}_\mathbb{C}$, a choice of 2-plane inside $\mathbb{R}^5$: $[V] \in \mathrm{Gr}_2(\mathbb{R}^5)$, and an element $\mathfrak{m}^{(k)} \in \mathfrak{t}_\mathbb{C}$ of the Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$.

Loosely speaking, we obtain the four-dimensional theory $S(\mathfrak{g}, C, D)$ by compactifying the six-dimensional theory $\mathfrak{X}_\mathfrak{g}$ over the surface C together with its defects:

$$S(\mathfrak{g}, C, D) := \mathfrak{X}_\mathfrak{g} // (C, D). \tag{4.3.1}$$

In this spirit the defects D should be thought of as providing boundary conditions at the punctures, which is necessary since C is not compact.

[4]By which we just mean that there are no defects/singularities included.

Remark 4.3.1. There is a difficulty arising by pure compactification that we have not discussed yet, namely the preservation of supersymmetry. Recall that to global symmetries G (such as supersymmetry, Poincaré symmetry and R-symmetry) one needs to choose a bundle with connection (P_G, ∇_G) such that the fields and operators are sections of associated bundles of P_G in certain representations. The symmetries preserved under a compactification are associated to covariantly constant quantities. However, there are generally no covariantly constant spinors in the associated spinor bundle to C. Witten ([Wit88],[Wit91]) introduced a procedure called *partial topological twisting* that circumvents this problem. We will not explain the general mechanics but describe briefly what it means for the situation at hand following [Moo12].

Recall that the super Lie algebra of the six-dimensional theory $\mathfrak{iso}(5,1) \oplus \mathfrak{so}(5)_R$ is the sum of the six-dimensional Poincaré algebra and the R-symmetry algebra. As explained in section 2.4 the metric must split and hence so must the structure algebra:

$$\mathfrak{so}(2)_{\mathrm{st}} \oplus \mathfrak{iso}(3,1) \hookrightarrow \mathfrak{iso}(5,1)$$

where the broken translations on C are not included anymore. Upon choosing a monomorphism

$$\mathfrak{so}(2)_R \oplus \mathfrak{so}(3)_R \hookrightarrow \mathfrak{so}(5)_R$$

and a principal Spin(2)-bundle $P_R \to C$ together with the Levi-Civita spin connection ∇_R on C, one can reduce the structure algebra of the $P_{\mathrm{spin}}(C) \times P_R$ to the diagonal subalgebra $\mathfrak{so}(2)_D \hookrightarrow \mathfrak{so}(2)_{\mathrm{st}} \oplus \mathfrak{so}(2)_R$. The super Lie algebra of symmetries of the resulting four-dimensional theory has thus even part

$$\mathfrak{g}_0 = \mathfrak{iso}(3,1) \oplus \mathfrak{so}(2)_D \oplus \mathfrak{so}(3)_R =: \mathfrak{iso}(3,1) \oplus \mathfrak{g}_R,$$

and is hence realized as a sum of four-dimensional Poincaré algebra and an R-symmetry algebra as expected. There are covariantly constant spinors under this reduced Lie algebra and there is an $\mathcal{N} = 2$ supersymmetry generated by the preserved supersymmetries of the descending six-dimensional generators.

Hence all the considerations from chapter 3 apply to the theories of class $\mathcal{S}$, in particular we recall that:

- The Coulomb branch is a Special Kähler manifold $\mathcal{B}$ of real dimension $2r$ (where r is the rank of $\mathfrak{g}$);
- There is a fibre bundle $\mathcal{M} \to \mathcal{B}$ with generic fibre a $2r$-torus (where r is the rank of $\mathfrak{g}$) which is singular over a codimension one locus $\mathcal{B}^{\mathrm{sing}} \subset \mathcal{B}$;
- The manifold $\mathcal{M}$ is Hyperkähler and is the Coulomb branch of the effective theory obtained by compactifying the four-dimensional theory on a circle;
- The projection $\mathcal{M} \to \mathcal{B}$ is holomorphic in two distinguished complex structures I and $-I$ of $\mathcal{M}$. We always define the $\mathbb{CP}^1$ worth of complex structures of $\mathcal{M}$ in such a way that the point 0 corresponds to I (and hence ∞ corresponds to $-I$).

In the next section we will see that in the particular case of the theories of class $\mathcal{S}$, the fibration $\mathcal{M} \to \mathcal{B}$ can be identified with the Hitchin integrable system which will yield a rich geometrical structure.

4.4 Connection to the Hitchin integrable system

There are two key insights from physics leading to the connection to the Hitchin integrable system:

1. The order of compactification from six to three dimensions is irrelevant for the resulting effective theory. Hence the Coulomb branch $\mathcal{M}$ of the three-dimensional theory can equivalently be obtained by compactifying from six dimensions first to five dimensions and then to three dimensions (with an appropriate topological twist). The reason both limits should agree is argued in [GMN13c] to be that the topological twisting preserves only quantities that are insensitive to the conformal scale of the surface C, hence there can not arise any relative length scales between C and S^1.

2. By the axioms of the theory $\mathfrak{X}$ the effective five-dimensional theory is a super Yang-Mills theory (SYM) with gauge group $\mathrm{SU}(K)$. It is argued in [GMN13c] that a certain combination of the adjoint scalars of the five-dimensional theory combine into a $(1,0)$-form φ on C in the twisted theory. More concretely, there is a principal $\mathrm{SU}(K)$-bundle $P \to C$ with unitary connection ∇, written locally as $\nabla = \mathrm{d} + A$ with gauge field $A = A_z \mathrm{d}z + A_{\bar{z}} \mathrm{d}\bar{z}$ cotangent to C, together with a "Higgs field" $\varphi \in \Gamma\left(C; \mathcal{K}_C \otimes \mathrm{ad}P\right)$. In terms of this gauge field A and the adjoint scalar field φ the BPS equations take the form

$$\begin{aligned} F_A + R^2[\varphi, \bar{\varphi}] &= 0, \\ \bar{\partial}_A \varphi := \mathrm{d}\bar{z}\left(\partial_{\bar{z}}\varphi + [A_{\bar{z}}, \varphi]\right) &= 0, \\ \partial_A \bar{\varphi} := \mathrm{d}z\left(\partial_z \bar{\varphi} + [A_z, \bar{\varphi}]\right) &= 0. \end{aligned} \tag{4.4.1}$$

   These equations are the well-known *Hitchin equations*, the moduli space of its solutions is called the *Hitchin space*. The physical insight is that the moduli space $\mathcal{M}$ of the three-dimensional theory is the space of 3d-Poincaré-invariant BPS configurations of the five-dimensional theory which can hence be identified with the Hitchin space [GMN13c].

Next, define the *spectral curve* for $u \in \mathcal{B}$

$$\Sigma_u := \left\{\lambda \in T^*C : \det\left(\lambda - \varphi_u\right) = 0\right\} \subset T^*C \tag{4.4.2}$$

with the determinant taken in the fundamental representation of $\mathfrak{su}(K)$. Note that for convenience we will often omit the basepoint $u \in \mathcal{B}$ in our notation. The equation can be spelled out to give

$$\lambda^K + \lambda^{K-1}\phi_1 + \cdots + \lambda\phi_{K-1} + \phi_K = 0, \tag{4.4.3}$$

where the ϕ_i are holomorphic i-differentials on C (i.e. sections of $\mathcal{K}_C^{\otimes i}$ where $\mathcal{K}_C$ is the canonical line bundle on C) with singularities at the puncture $\mathfrak{s}_j$ prescribed by the defect D_j. In the easiest case of simple poles with regular semisimple residue ϕ_i has a pole of order i at each singularity with leading term in local coordinates near $\mathfrak{s}_j$

$$\phi_i \sim \frac{e_i(\mathfrak{m}^{(j)})}{z^i}(\mathrm{d}z)^i + \ldots \tag{4.4.4}$$

where e_i denotes the elementary symmetric function. Note that the subleading behaviour depends on the choice of base point $u \in \mathcal{B}$. More general singularities are covered systemetically in sections 3 and 4 of [GMN13c]. In any case, $\mathcal{B}$ is parametrized by the differentials $(\phi_1, \dots, \phi_K)$ and since these have prescribed behaviour at the singular points and are holomorphic elsewhere, the space $\mathcal{B}$ is a torsor (i.e. a principal homogeneous space) for the vector space

$$\bigoplus_{j=1}^{K} H^0\left(\overline{C}; \mathcal{K}_{\overline{C}}^{\otimes j} \otimes \mathcal{O}\left(-\Sigma_{k=1}^{n}(j-1)\mathfrak{s}_k\right)\right)$$

with $\overline{C}$ the compact Riemann surface obtained by filling in the punctures of C. This sheds light on the fibre bundle $\mathcal{M} \to \mathcal{B}$ in the following way: We know from physics that $\mathcal{B}$ is parametrized by the Casimirs of φ ([GMN13c]) which we have just seen to be polynomials in $\mathrm{tr}(\varphi^k)$, hence the fibration takes the form

$$(A, \varphi) \to \{\text{Casimirs of } \varphi\}.$$

This map is known as the *Hitchin fibration* with generic fiber an abelian variety known as the Prym variety of the projection $\bar{\Sigma}_u \to C$. In this way $\Sigma \to C$ is identified as a branched K:1-covering and Σ inherits a canonical 1-form λ by restriction of the Liouville 1-form of T^*C. In physics, Σ is known as the *Seiberg-Witten curve* and λ is the corresponding *Seiberg-Witten differential*. For generic $u \in \mathcal{B}$ the branch points of $\Sigma_u \to C$ are all simple but on the special sublocus $\mathcal{B}^{\text{sing}}$ two different branch points collide corresponding to the fact that (4.4.3) has multiple roots.

The spectral curve allows for neat geometric interpretations of many aspects of the theory that we have covered in chapter 3. For example, the local system of charge lattices over the four-dimensional Coulomb branch $\mathcal{B}$ can be interpreted as follows: The cover $\pi : \Sigma_u \to C$ gives rise to a lattice

$$\Gamma_u := \ker(\pi_*) \subset H_{1,cpct}(\Sigma_u, \mathbb{Z}) \tag{4.4.5}$$

that glues to a local system of lattices Γ on $\mathcal{B}^* := \mathcal{B} \backslash \mathcal{B}^{\text{sing}}$. Note that we take compact homology since neither C nor Σ are compact with the punctures removed. The symplectic form $\langle \cdot, \cdot \rangle$ is then given by the intersection pairing of compactly supported 1-cycles. The pairing generally has a radical Γ_{f} (called *flavour charges*) represented by loops around the punctures and Γ is an extension of the local system of gauge charges Γ_g by Γ_{f}:

$$0 \to \Gamma_{\text{f}} \to \Gamma \to \Gamma_g \to 0. \tag{4.4.6}$$

In this spirit, the fibre $\Gamma_{g,u}$ of gauge charges is simply $\Gamma_{g,u} = H_1(\bar{\Sigma}_u, \mathbb{Z})$. The central charge function $Z(u) \in \mathrm{Hom}(\Gamma_u, \mathbb{C})$ can be reinterpreted as

$$Z_\gamma(u) = \frac{1}{\pi} \oint_\gamma \lambda. \tag{4.4.7}$$

Before we will give the interpretation of BPS states in this picture in the next section, we want to point out an important difference between the lattices Γ_g and Γ_f, namely that while Γ_f is fibered trivially over $\mathcal{B}^*$, the gauge charges of Γ_g generally pick up a monodromy under parallel transport along a non-trivial loop in $\mathcal{B}^*$. In particular, the above splitting of Γ is not globally well-defined and around a non-trivial loop in $\mathcal{B}^*$ the charge $\gamma = \gamma_g + \gamma_f$ is subject to monodromies

$$\begin{aligned} \gamma_g &\mapsto \gamma_g + M(\gamma_g), \quad & M \in \ \mathrm{Aut}(\Gamma_g), \\ \gamma_f &\mapsto \gamma_f + N(\gamma_g), \quad & N \in \ \mathrm{Hom}(\Gamma_g, \Gamma_f). \end{aligned} \tag{4.4.8}$$

The symplectic pairing $\langle \cdot, \cdot \rangle$ is monodromy invariant and the restriction of the globally defined section $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ (the *central charge*) to Γ_f is constant on $\mathcal{B}^*$, i.e. there is some $m \in \Gamma_f^* \otimes_{\mathbb{Z}} \mathbb{C}$ which is independent of $u \in \mathcal{B}^*$ such that $Z_{\gamma_f} = m \cdot \gamma_f$.

4.5 String webs: The geometry of BPS states

In this section we are interested in the BPS particles of the 4d theory. These descend from higher-dimensional BPS states in six dimensions, namely from BPS strings. How do these arise? Recall that there are two types of branes in M-theory: M5-branes and M2-branes. We have stated that the six-dimensional theory of type A_{K-1} arises as the low energy theory of K parallel M5-branes in the absence of gravity (i.e. in the limit where gravity decouples). Now we have considered the special case where the space time is of the form $C \times \mathbb{R}^{3,1}$ such that the M5-branes wrap around the surface C. In this picture, the lifts to the K-fold cover Σ correspond to the K different branes.

Next, we add $M2$ branes with boundary on two parallel M5-branes i and j. These intersections are necessarily two-dimensional and we take them to be the product of a 1-manifold (a *string*) in C and a 1-manifold on $\mathbb{R}^{1,3}$. Moving to the cover $\Sigma \to C$, the former string ascends to a pair of strings, one on each the i-th and the j-th sheet of the cover. We will explain this following [GMN13b].

Having an oriented segment of a string of this type passing through a point $z_0 \in C$ we pick two solutions $\lambda^{(i)}, \lambda^{(j)}$ $(i \neq j)$ of equation (4.4.2) which lie on the i-th resp. j-th sheet of the lift of a neighbourhood of $z_0 \in C$ to Σ (we will keep the basepoint $u \in \mathcal{B}$ fixed). The choice of these sheets is very important in what follows and we will henceforth call such a string an ij-string, while the reversal of the orientation of the string yields a ji-string. In analogy to (4.4.7) the central charge of an ij-string σ is given by the integral

$$Z_\sigma(u) = \frac{1}{\pi} \int_\sigma (\lambda^{(i)} - \lambda^{(j)}). \tag{4.5.1}$$

Similarly one can define its mass by

$$M_\sigma(u) = \frac{1}{\pi} \int_\sigma |\lambda^{(i)} - \lambda^{(j)}|. \tag{4.5.2}$$

We call such a string BPS if it satisfies the BPS equation $|Z| = M$ which we see holds if the 1-form $\lambda^{(i)} - \lambda^{(j)}$ has constant phase along σ. What does that mean? By introducing a local coordinate

$$w^{ij}(z) := \int_{z_0}^{z} (\lambda^{(i)} - \lambda^{(j)}) \tag{4.5.3}$$

in an open neighbourhood of $z_0 \in C$ we can define an *ij-trajectory of phase ϑ* to be a line along which

$$\mathrm{Im}(e^{-i\vartheta} w^{ij}) = 0. \tag{4.5.4}$$

This gives a local foliation of the surface by straight lines in the w^{ij}-plane. The ij-trajectories come equipped with a natural orientation by pointing in the direction where $\mathrm{Re}(e^{-i\vartheta} w^{ij})$ increases. As for the ij-strings, reversal of orientations means exchanging

i and j which gives a ji-trajectory. In any case, a *BPS string of phase* ϑ is an ij-string stretched along an ij-trajectory of phase ϑ.

What happens as we follow along the string? It can end in two different ways: Either on an (ij) branch point where $\lambda^{(i)} - \lambda^{(j)} = 0$ and the i-th and j-th sheet of the cover collide, or in a junction. A 3-point junction is where an ij-, a jk- and a ki-string meet and consequently

$$(\lambda^{(i)} - \lambda^{(j)}) + (\lambda^{(j)} - \lambda^{(k)}) + (\lambda^{(k)} - \lambda^{(i)}) = 0.$$

The combined web of these strings is BPS if all the strings involved are BPS of *the same phase* ϑ. We are interested in *finite* webs:

Definition 4.5.1. A *finite string web* on a punctured Riemann surface C is a connected graph whose segments consist of BPS strings and whose vertices consist of junctions such that the end points of the graph lie on the branch points.

Now take an ij-string stretched along an oriented path p on C, there is a canonical lift p_Σ to Σ as the union of the lift $p^{(i)}$ to the i-th sheet and the lift $-p^{(j)}$ to the j-th sheet with reversed orientation. Doing that for all strings in a finite web N and joining the lifts yields a 1-cycle N_Σ on Σ with homology class $[N_\Sigma] \in H_1(\Sigma;\mathbb{Z})$ which is the charge of the BPS state. The central charge of a finite string web N is just the sum of the central charges of the strings which depends only on the homology class $\gamma := [N_\Sigma]$ and is hence given by

$$Z_\gamma = \frac{1}{\pi}\oint_\gamma \lambda$$

in accordance with (4.4.7). As we have seen the string webs encode BPS states of the four-dimensional theory with $\mathcal{N} = 2$ charge Z_γ which hence have BPS degeneracies $\Omega(u,\gamma)$ that are expected to satisfy the KSWCF (3.3.17). However, string webs are a non-generic case: for generic value of ϑ there is no string web. To shed some light on this, let us describe the foliation of C by straight lines in the $w := w^{ij}$-plane which we will henceforth call *WKB foliation* resp. *WKB curves*. Due to the definition of w in (4.5.3) we need to study the behaviour of the differential λ near singularities $\mathfrak{s}_k$. In accordance with (4.4.4) a regular singularity would admit a pole of the form

$$\lambda_i = \mathfrak{m}_i^{(k)}\frac{\mathrm{d}z}{z} + \mathrm{reg} \tag{4.5.5}$$

in a local coordinate with $z = 0$ at $\mathfrak{s}_j$. The differentials are locally of the form $\lambda_i = f_i(z)\mathrm{d}z$. Fix a parametrization of the WKB curve (which we call a *WKB path*), then the differentials must satisfy

$$(f_i(z(t)) - f_j(z(t)))\,\frac{\mathrm{d}z}{\mathrm{d}t} = e^{i\vartheta}. \tag{4.5.6}$$

Therefore, the WKB path must asymptote to

$$z(t) = z_0 \exp\left(\frac{e^{i\vartheta}t}{\mathfrak{m}_i^{(k)} - \mathfrak{m}_j^{(k)}}\right) \tag{4.5.7}$$

which parametrizes a spiral for generic values of ϑ and masses $\mathfrak{m}_i^{(k)}, -\mathfrak{m}_j^{(k)}$. Notice that this induces a natural ordering of the sheets of the cover $\Sigma \to C$ in a neighbourhood of the singularity $\mathfrak{s}_k$ by demanding that $i < j$ when the WKB path spirals into the singularity. In particular, the area around the singularity serves as a basin of attraction that captures all the trajectories. The behaviour of the WKB paths is very different around irregular singularities, i.e. singularities $\mathfrak{s}_k$ around which the differential has a higher pole

$$\lambda \sim \frac{\mathrm{d}z}{z^r} + \dots \tag{4.5.8}$$

with $r > 1$. We will not dwell on making this more precise here. Instead, we are interested in the behaviour near an (ij)-branch point $\mathfrak{b}$. For generic $u \in \mathcal{B}^*$ it is a simple branch point, i.e. a simple zero of the discriminant $\prod_{a<b}(\lambda_a - \lambda_b)^2$, hence allowing to write in a local coordinate z

$$\lambda_i = (c + \sqrt{z} + \dots)\mathrm{d}z, \qquad \lambda_i = (c - \sqrt{z} + \dots)\mathrm{d}z \tag{4.5.9}$$

with some constant c. The local coordinate w^{ij} must therefore asymptote to

$$w^{ij}(z(t)) = \int_{\mathfrak{b}}^{z(t)} (\lambda_i - \lambda_j) \sim \frac{4}{3} z(t)^{3/2} \overset{!}{=} e^{i\vartheta} t. \tag{4.5.10}$$

This can be explicitly solved to

$$z(\tau) = \tau e^{2i\vartheta/3} \tag{4.5.11}$$

with $\tau = t^{2/3}$. Thus any simple branch point is the origin of exactly three trajectories. This is the key insight that allows for a duality to certain triangulations on C in the case $\mathfrak{g} = \mathfrak{su}(2)$ which we will explain in the next section. First, we want to quickly explain the global behaviour of WKB curves.

Definition 4.5.2. A WKB curve is called

- *finite* if it is closed or if it has both ends on a branch point or a three point junction;
- *generic* if both ends asymptote to a singular point (and possibly to the same one);
- *separating* if one end asymptotes to a singular point while the other end is on a branch point;
- *divergent* if it is not closed and if at least one of the ends does not approach any limit.

Let us quickly explain the origin of this nomenclature. The end of a WKB curve should generically asymptote to a singular point due the neighbourhood of such a point acting as a basin of attraction. On the other hand, a WKB curve for which one end asymptotes to a singular point cannot have finite length, hence a finite WKB curve is one which has finite total length. The separating WKB paths span borders between generic WKB paths of different homotopy type because generic WKB paths come in 1-parameter families of homotopy equivalent curves.

We will restrict our attention to the case $\mathfrak{g} = \mathfrak{su}(2)$ in the next section where the functions $\mathcal{X}_\gamma$ can be derived from Fock-Goncharov coordinates on $\mathcal{M}$ associated to certain triangulations of C. As we will see there is a canonical triangulation $T_{\mathrm{WKB}}(\vartheta, \lambda^2)$ which is determined by the choice of the quadratic differential λ^2 and a phase ϑ.

4.6 The $\mathfrak{sl}(2,\mathbb{C})$ case

In the case of $\mathfrak{g} = \mathfrak{su}(2)$ the situation simplifies tremendously because now there are only two sheets in the covering $\Sigma \to C$ and correspondingly there cannot be any 3-string junctions. Hence there are only two possible topologies for finite strings (because they are always connected by definition), namely

1. A closed loop which represents a vectormultiplet and should hence contribute a BPS index $\Omega(u,\gamma) = 1$;

2. A saddle connection is given by a straight line in the w-plane that connects two branch points on C. Upon the lift to the cover Σ this becomes a 1-cycle that consists of two strings connecting the two branch points, one on each sheet and of opposite orientation. This corresponds to a hypermultiplet in the 4d theory and should hence have BPS index $\Omega(u,\gamma) = -2$.

In the absence of finite WKB curves it was shown in [GMN13c] that any WKB curve is either generic or separating (recall that we always assume that there is at least one singular point). It is well-known [Str84] that the generic WKB curves come in 1-parameter families sweeping out cells that are bounded by unions of separating WKB curves. Because there are no finite WKB curves, each boundary component can contain only one branch point, hence there are two possible topologies for a cell determined by whether those two branch points are distinct or if they are the same. Upon fixing λ^2 and ϑ one obtains a canonical triangulation of C in the following way: Choose one generic WKB path in every cell, this determines a decomposition of C into faces such that every face contains one branch point. As we have seen around (4.5.11) every branch point emits exactly three trajectories whose other ends are at corners of the face. Hence all the faces must be triangles (possibly degenerate) which determines the triangulation of C. The topology of this type does not depend on the choice of the representative of generic WKB path in every 1-parameter family, thus by "canonical triangulation" we really mean a triangulation of canonical topological type and we will henceforth mean isotopy classes of triangulations when we speak of triangulations. So far we have defined an ideal triangulation $T_{\text{WKB}}(\vartheta,\lambda^2)$ on C whose vertices are the singular points $\mathfrak{s}_k$.[5] Next we need to choose a certain decoration of this triangulation which is done as follows.

Define a connection on C via

$$\mathcal{A} = \mathcal{A}_\zeta := \frac{R}{\zeta}\varphi + A + R\zeta\bar{\varphi} \tag{4.6.1}$$

with A the gauge field and φ the Higgs field. The Hitchin equations (4.4.1) are equivalent to saying that $\mathcal{A}$ is flat for all choices of $\zeta \in \mathbb{C}^\times$. In the case of regular singularities the decoration corresponds to picking one of the two eigenlines of the monodromy around each of the singular points $\mathfrak{s}_i$ which is equivalent to picking a flat section s_i (up to a constant scale) satisfying $(\mathrm{d} + \mathcal{A})s_i = 0$ for each singular point. This decoration allows for the definition of coordinates $\mathcal{X}_E^T$ labelled by the triangulation $T = T_{\text{WKB}}(\vartheta,\lambda^2)$ and an edge $E \subset T$ due to the work of Fock and Goncharov [FG06] in the following way: E is edge of two triangles which together form a quadrilateral Q_E (we omit the degenerate case where some of the vertices collide). The vertices of the quadrilateral are singular points $\mathfrak{s}_i$ that we label counter-clockwise to be points $\mathcal{P}_1,\ldots,\mathcal{P}_4$ such that $\mathcal{P}_1$ and $\mathcal{P}_3$ are the vertices at the ends of E. Note that this labelling

[5] Since λ^2 depends on the choice of a base point $u \in \mathcal{B}$, we frequently denote the triangulation by $T_{\text{WKB}}(\vartheta, u)$ if we want to put emphasis on the base point.

is only well-defined up to a simultaneous exchange $\mathcal{P}_1 \leftrightarrow \mathcal{P}_3$ and $\mathcal{P}_2 \leftrightarrow \mathcal{P}_4$ which will not effect the coordinate $\mathcal{X}_E^T$. The decoration of $T_{\text{WKB}}(\vartheta, \lambda^2)$ gives flat sections $s_1, \dots, s_4$ at $\mathcal{P}_1, \dots, \mathcal{P}_4$, respectively, that allow the definition

$$\mathcal{X}_E^T := -\frac{(s_1 \wedge s_2)(s_3 \wedge s_4)}{(s_4 \wedge s_1)(s_2 \wedge s_3)} \tag{4.6.2}$$

with all s_i evaluated at the same point $\mathcal{P} \in Q_E$. Note that the ambiguity of the s_i by rescaling cancel out in the quotient and that the choice of $\mathcal{P}$ is unambiguous due to the flat connection $\mathcal{A}$ taking values in $\mathfrak{sl}(2, \mathbb{C})$.

Next we want to explain how these induce Darboux coordinates $\mathcal{X}_\gamma^{\text{RH}}$ labelled by a choice of $\gamma \in \Gamma$. Start by fixing an edge E of the triangulation of C, the corresponding quadrilateral Q_E consists of two triangles with exactly one branch point in the interior of each. We will only explain what happens when those are distinct branch points, for the degenerate case one needs to pass to a certain covering surface where the degeneration is resolved and one can apply a similar technique that we explain for the non-degenerate case. Upon choosing the edge E one can draw a loop inside Q_E around the two branch points which we may lift to a connected loop on the cover Σ which in turn defines a homology class $\gamma_E^\vartheta \in H_1(\Sigma; \mathbb{Z})$.

This is well-defined modulo two choices: the choice of an orientation for the lifted loop and the choice of a sheet of the cover to which the loop is lifted. Each of those choices enters the homology class through an overall sign. This is clear for the choice of an orientation and can be seen for the two lifts to the different covers by the action of deck transformations. One fixes this ambiguity by first choosing the orientation of the lift $\hat{E}$ in such a way that the positively oriented tangent vector ∂_t of $\hat{E}$ obeys $e^{-i\vartheta}\lambda \cdot \partial_t > 0$. $\hat{E}$ defines a cycle in the relative homology $H_1(\Sigma, \{\mathfrak{s}_k\}; \mathbb{Z})$ which has a well-defined intersection pairing $\langle \cdot, \cdot \rangle$ with $H_1(\Sigma; \mathbb{Z})$. Finally, it is demanded that $\langle \gamma_E^\vartheta, \hat{E} \rangle = 1$ which is independent of the choice of a lift of the edge and hence fixes the ambiguous sign.

This procedure yields well-defined vectors $\gamma_E^\vartheta \in H_1(\Sigma; \mathbb{Z})$. It is shown in [GMN13c] that the entire lattice $\hat{\Gamma}$ is in fact generated by the set $\{\gamma_E^\vartheta\}_E$ for fixed ϑ and E running over all edges of the triangulation $T_{\text{WKB}}(\vartheta, \lambda^2)$. We can thus define functions $\mathcal{X}_\gamma^{\vartheta,u}$ for $\gamma \in \hat{\Gamma}$, $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ and $u \in \mathcal{B}$ by demanding

$$\mathcal{X}_{\gamma_E}^{\vartheta,u} := \mathcal{X}_E^{T_{\text{WKB}}(\vartheta,u)} \text{ for all edges } E \text{ of } T_{\text{WKB}}(\vartheta, u) \text{ and} \tag{4.6.3}$$

$$\mathcal{X}_{\gamma_1}^{\vartheta,u}\mathcal{X}_{\gamma_2}^{\vartheta,u} = \mathcal{X}_{\gamma_1+\gamma_2}^{\vartheta,u} \text{ for all } \gamma_1, \gamma_2 \in \hat{\Gamma}. \tag{4.6.4}$$

The second condition resembles (3.6.2) but the question remains how these functions $\mathcal{X}_\gamma^{\vartheta,u}$ relate to the functions $\mathcal{X}_\gamma^{\text{sf}}$ and $\mathcal{X}_\gamma^{\text{RH}}$ and in which way they satisfy the properties in section 3.6. In order to explain this, we must first define the half-plane centred at the ray $e^{i\vartheta}\mathbb{R}_+$ by

$$\mathbb{H}_\vartheta := \left\{ \zeta \in \mathbb{C}^\times : |\arg(\zeta) - \vartheta| < \frac{\pi}{2} \right\}. \tag{4.6.5}$$

As before, fix an open set $U \subset \mathcal{B}$ over which Γ is trivializable together with a base point $u_0 \in U$ and a quadratic refinement $\sigma : \Gamma \to \{\pm 1\}$. Then the $\mathcal{X}_\gamma^{\vartheta,u_0}$ form a family of $\mathbb{C}^\times$-valued functions on $\pi^{-1}(U) \times \mathbb{H}_\vartheta$ satisfying the following properties (recall that π is the bundle projection $\pi : \mathcal{M} \to \mathcal{B}$) as shown in [GMN13c]:

1. $\mathcal{X}_\gamma^{\vartheta,u_0}(\cdot; \zeta) = \overline{\mathcal{X}_{-\gamma}^{\vartheta+\pi,u_0}(\cdot; -\overline{1/\zeta})}$ (which resembles (3.6.3);

2. for fixed $\zeta \in \mathbb{H}_\vartheta$, $\mathcal{X}_\gamma^\vartheta(\cdot;\zeta)$ is holomorphic in the complex structure $J^{(\zeta)}$ on $\mathcal{M}$ (resembling the Cauchy-Riemann equations (3.6.4) on $\mathcal{M}$);[6]

3. for fixed $(u,\theta) \in \pi^{-1}(U)$ the assignment $\zeta \mapsto \mathcal{X}_\gamma^\vartheta(u,\theta;\zeta)$ is holomorphic for $\zeta \in \mathbb{H}_\vartheta$ (similar to property (D) in section 3.6);

4. the Poisson bracket associated to the symplectic structure (compare to (E) and (F) in section 3.6) is given after rescaling by a factor of $4\pi^2 R$ by

$$\{\mathcal{X}_{\gamma_1}, \mathcal{X}_{\gamma_1}\} = \langle\gamma_1,\gamma_2\rangle \mathcal{X}_{\gamma_1}\mathcal{X}_{\gamma_2}; \tag{4.6.6}$$

5. the following limit exists for $\zeta \in \mathbb{H}_\vartheta$:

$$\lim_{\zeta\to 0} \mathcal{X}_\gamma^{\vartheta,u_0}(u_0,\theta;\zeta)\exp\left(-\zeta^{-1}\pi R Z_\gamma(u_0)\right) \tag{4.6.7}$$

   and similarly for $\zeta \to \infty$ due to property 1, hence yielding condition (G) from section 3.6;

6. in the large R limit, $\mathcal{X}_\gamma$ approach $\mathcal{X}_\gamma^{\text{sf}}$ to exponentially small deviations, i.e.

$$\mathcal{X}_\gamma = \mathcal{X}_\gamma^{\text{sf}}\left(1 + \mathcal{O}(e^{-\text{const.}\cdot R})\right) \tag{4.6.8}$$

   in regions bounded away from the singular points of $\mathcal{B}$;

7. as a function of (ϑ, u_0), $\mathcal{X}_\gamma^{\vartheta,u_0}$ is piecewise constant with jumps at pairs (ϑ_0, u_0) for which there exists a γ_{BPS} with $\arg Z_{\gamma_{\text{BPS}}}(u_0) = -\vartheta$ and $\Omega(u_0,\gamma_{\text{BPS}}) \neq 0$. More precisely, let $\mathcal{K}_{\gamma_{\text{BPS}}}$ be holomorphic Poisson transformations acting on the coordinates $\mathcal{X}_\gamma$ by

$$\mathcal{K}_{\gamma_{\text{BPS}}}(\mathcal{X}_\gamma) := \mathcal{X}_\gamma\left(1 - \sigma(\gamma_{\text{BPS}})\mathcal{X}_{\gamma_{\text{BPS}}}\right)^{\langle\gamma,\gamma_{\text{BPS}}\rangle} \tag{4.6.9}$$

   and let their product for aligned charges be

$$\mathbf{S}_{\vartheta_0,u_0} := \prod_{\gamma_{\text{BPS}}\colon \arg(Z_{\gamma_{\text{BPS}}}(u_0)) = -\vartheta_0} \mathcal{K}_{\gamma_{\text{BPS}}}^{\Omega(u_0,\gamma_{\text{BPS}})}. \tag{4.6.10}$$

   Note that as in (3.6.8) we do not need to order the product because the charges are aligned by definition. The statement is that upon reaching the discontinuity at (ϑ_0, u_0) the Darboux functions $\mathcal{X}_\gamma^{\vartheta,u_0}$ jump according to

$$\lim_{\vartheta\to\vartheta_0^+} \mathcal{X}_\gamma^{\vartheta,u_0} = \mathbf{S}_{\vartheta_0,u_0}\left(\lim_{\vartheta\to\vartheta_0^-} \mathcal{X}_\gamma^{\vartheta,u_0}\right). \tag{4.6.11}$$

Hence we have explicitly given a construction for coordinates $\mathcal{X}_\gamma^\vartheta$ on $\mathcal{M} \times \mathbb{H}_\vartheta$ satisfying the conditions (A)-(G) from section 3.6 modulo the detail that the parameter ζ is only defined on the half-plane $\mathbb{H}_\vartheta$ rather than on $\mathbb{C}^\times$. This is due to a Stokes phenomenon which we will not explain here in detail but has no influence on the result that these functions $\mathcal{X}_\gamma^\vartheta$ are holomorphic Darboux coordinates. Consequently, one is able to explicitly compute a Hyperkähler metric for $\mathcal{M}$ by the arguments of [GMN10]. However,

[6] We will frequently omit the superscripts of $\mathcal{X}_\gamma^{\vartheta,u_0}$ and lighten the notation to $\mathcal{X}_\gamma^\vartheta$ or even $\mathcal{X}_\gamma$ if we do not want to emphasize the dependence on u_0 resp. (ϑ, u_0), or if the statement holds for these in a uniform way.

the construction given is sensible at all values of R (though it is to my best knowledge currently unknown if they remain pole-free for small R) and is hence preferable to the coordinates $\mathcal{X}_\gamma^{\mathrm{RH}}$ obtained from a Riemann-Hilbert problem, at least for the theories of class $\mathcal{S}$. In fact, the latter are easily obtained from the $\mathcal{X}_\gamma^\vartheta$ by specializing $\vartheta = \arg(\zeta)$.

There is one last thing worth dwelling on, namely the interpretation of the wall-crossing behaviour in terms of the WKB triangulations $T_{\mathrm{WKB}}(\vartheta, u)$. As explained in some length in [GMN13c], upon changing ϑ the triangulation remains its topological type around generic (ϑ, u) but changes its type when ϑ is the inclination of some BPS ray $l_{\gamma,u}$ (defined in (3.3.15)). At those points, the decorated triangulation undergoes *flips* (that is within one or more quadrilaterals the diagonal "flips" while the decoration stays the same), *pops* (where the triangulation is invariant but the decoration is changed by swapping the two eigenlines at a vertex) and *juggles* (these relate different limit triangulations which we have not discussed because we skipped degenerate edges). It is known for Fock-Goncharov coordinates and translates to the Darboux coordinates considered here that these are acted on by Kontsevich-Soibelman symplectomorphisms $\mathcal{K}_{\gamma_{\mathrm{BPS}}}$ related to BPS states at which the triangulations transform. It was shown in [GMN13c] that these symplectomorphisms exhibit the KS WCF with the correct BPS degeneracies Ω given by (4.6.11). This sheds new light on the discussion in the previous section that BPS states should be presented by *finite* WKB curves which in the A_1 case are either closed loops (vectormultiplets) or saddle connections (hypermultiplets).

Chapter 5

Spectral Networks

In this final chapter we aim to explain how WKB curves behave for $\mathfrak{su}(K)$-theories of class $\mathcal{S}$ and $K \geq 3$. The key new ingredient comes from physics and has been touched in our discussion of the theory $\mathfrak{X}$: There are surface defects associated to a 2-manifold inside the six-dimensional theory (and some other data). Putting the 2-manifold at a fixed linear subspace of Minkowski space, the surface defects $\mathbb{S}_z$ come canonically with the choice of a point $z \in C$. Varying this point along a path $\wp$ yields a *supersymmetric interface* $L_{\wp,\vartheta}$ (the dependence on the phase ϑ is spelled out in section 5.2) which is again a surface defect of the six-dimensional theory: It arises as the product of a path $\wp$ on C and a boundary between two 2-manifolds in Minkowski space, thus a line operator in the four-dimensional theory. We hence obtain a family of two-dimensional theories sitting inside the four-dimensional theory of class $\mathcal{S}$ which come naturally in a smooth modulus C and with a family of line operators.

Here is the outline for this chapter: We begin in section 5.1 by reviewing the KSWCF and explaining a similar phenomenon occuring in two-dimensional QFTs with $\mathcal{N} = (2,2)$ supersymmetry due to Cecotti and Vafa [CV93]. These wall-crossing formulas can be combined into a 2d-4d WCF whose abstract statement we give following [GMN12]. We start explaining the physical setting in which this appears in section 5.2 and will see that it is naturally the extension of the theories of class $\mathcal{S}$ explored so far by adding *defects* the way we have explained above. These enhanced theories naturally come with additional indices μ and ω which should be thought of as counting distinguished states in the two-dimensional theory resp. the coupled 2d-4d system.

We continue to describe the concepts of *Formal Parallel Transport* $F(\wp, \vartheta)$, a function depending on a path $\wp \subset C$ and the phase ϑ, as well as *Spectral Networks* $\mathcal{W}_\vartheta$, a collection of WKB curves on C, in section 5.3. Both of these are defined in terms of μ and ω. The key statement is the Formal Parallel Transport Theorem (Theorem 5.3.5 below) that explains that μ and ω are fixed by requiring certain properties $F(\wp, \vartheta)$ and $\mathcal{W}_\vartheta$. This includes in particular a Wall-Crossing Formula which we will identify with the 2d-4d WCF! To be more precise, we will see in section 5.4 that for generic ϑ these jumps happen for the 2d indices μ and represent the birth (or death) of 2d solitons. Slowly varying ϑ does not change the topology of the Spectral Networks but for certain critical values ϑ_c the topology suddenly jumps. This usually happens when two walls meet head-on, combining into a two-way street as pictured in figure 5.4. Section 5.5 aims at explaining that the jumps in these cases represent a jump in the 4d theory and hence the generation or annihilation of a 4d BPS state.

5.1 2d-4d Wall-Crossing

In this section we want to describe a wall-crossing formula for so called *2d-4d systems*. Roughly speaking, these consist of both a two-dimensional and a four-dimensional theory which impact each other and are somewhat "intertwined". We will describe them in more detail in the following section. Their wall-crossing behaviour was introduced in [GMN12] and can be made mathematically precise which is the aim of this section. Since they combine features from wall-crossing formulas in four dimensions and in two dimensions, we will start by quickly reviewing these (following [GMN12]).

Wall-crossing for 4d $\mathcal{N}=2$ theories

The KSWCF has already been described in section 3.3 so we will keep it brief. We fix $u \in \mathcal{B}$ in this section and will suppress it in our notation. Starting point is the charge lattice Γ (suppressing the subscript u) together with a bilinear antisymmetric pairing $\langle \cdot, \cdot \rangle$ taking values in the integers, a linear central charge function $Z : \Gamma \to \mathbb{C}$ and a set of *BPS degeneracies* $\Omega : \Gamma \to \mathbb{Z}$. We are mainly interested in the functions Ω which depend piecewise constantly on Z with jumps where the phases of two central charges Z_{γ_1} and Z_{γ_2} become aligned.

This jump is captured by the wall-crossing formula which is handily described by first introducing formal variables $X_\gamma \in \mathbb{C}^\times$ for $\gamma \in \Gamma$ obeying

$$X_{\gamma_1} X_{\gamma_2} = (-1)^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1+\gamma_2}. \tag{5.1.1}$$

Note that this differs from (3.3.8) because we have absorbed the quadratic refinement (3.3.12) into the definition for later convenience. These variables are acted on by the *$\mathcal{K}$-factors*

$$\mathcal{K}^{\Omega}_{\gamma_1} : X_{\gamma_2} \mapsto (1 - X_{\gamma_1})^{\langle \gamma_1, \gamma_2 \rangle \Omega(\gamma_1)} X_{\gamma_2} \tag{5.1.2}$$

where we have made two changes compared to (3.3.14): Firstly, we swapped the two factors which will be absorbed by the opposite ordering of factors in the wall-crossing formula. Secondly and more severely, we have absorbed the BPS degeneracy $\Omega(\gamma)$ into the definition because the maps $\mathcal{K}_\gamma$ do not allow for a natural extension to the 2d-4d case that we will describe below. A BPS ray is a ray

$$\ell_\gamma = Z_\gamma \mathbb{R}_- \subset \mathbb{C} \tag{5.1.3}$$

which we assign to any charge γ with $\Omega(\gamma) \neq 0$. Now choose an angular convex sector $\sphericalangle \subset \mathbb{C}$ with apex at the origin and define

$$A(\sphericalangle) := \prod^{\curvearrowleft}_{\gamma : \ell_\gamma \subset \sphericalangle} \mathcal{K}^{\Omega}_{\gamma} \tag{5.1.4}$$

where the arrow above the product sign means that the ordering is to be taken in counter-clockwise order of the rays. Then the KSWCF (Theorem 3.3.1) states that $A(\sphericalangle)$ remains invariant under variation of Z provided that no BPS ray crosses the boundary of $\sphericalangle$.

Wall-crossing for 2d $\mathcal{N}=(2,2)$ theories

A similar wall-crossing formula has been found by Cecotti and Vafa for two-dimensional QFTs with $\mathcal{N}=(2,2)$ supersymmetry [CV93]. The basic data entering are

- a finite set $\mathcal{V}$ (whose elements will label vacua in a (1+1)-dimensional QFT),

- a central charge function $Z : \mathcal{V} \to \mathbb{C}$ and
- degeneracies $\mu_{ij} \in \mathbb{Z}$ labelled by pairs $i \neq j \in \mathcal{V}$.

As in the 4d case, the degeneracies μ_{ij} are piecewise constant as functions of Z that jump when Z crosses a certain wall with the jumping behaviour expressed by the Cecotti-Vafa Wall-Crossing Formula (CVWCF). These walls are called "walls of marginal stability" and arise when three charges Z_i, Z_j, Z_k become collinear as points of $\mathbb{C}$.

We begin similarly to the 4d setting with the introduction of formal variables X_i and X_{ij} labelled by $i, j \in \mathcal{V}$ with a left-multiplication of the latter on the former given by

$$X_{ij}X_k := \delta_{jk}X_i. \tag{5.1.5}$$

The CVWCF is expressed in terms of *$\mathcal{S}$-factors* $\mathcal{S}^{\mu}_{ij}$ acting on the variables X_k via

$$\mathcal{S}^{\mu}_{ij} : X_k \mapsto (1 - \mu_{ij}X_{ij})\, X_k = X_k - \delta_{jk}\mu_{ij}X_i \tag{5.1.6}$$

with the BPS rays ℓ_{ij} assigned to pairs (i, j) with $\mu_{ij} \neq 0$ given by

$$\ell_{ij} := Z_{ij}\mathbb{R}_- \subset \mathbb{C} \tag{5.1.7}$$

where $Z_{ij} := Z_i - Z_j$. Choose a convex sector $\sphericalangle \subset \mathbb{C}$ as in the 4d case and define

$$A(\sphericalangle) := \prod^{\curvearrowleft}_{i,j:\ell_{ij}\subset\sphericalangle} \mathcal{S}^{\mu}_{ij} \tag{5.1.8}$$

with order taken by increasing argument of the corresponding BPS rays. The CVWCF states that $A(\sphericalangle)$ is constant under variation of Z as long as no BPS ray crosses the boundary of $\sphericalangle$.

Wall-crossing for 2d-4d coupled systems

The 2d-4d WCF involves data from both the 2d and the 4d WCF put together in a certain fashion. The starting point are again a finite set $\mathcal{V}$ and a lattice Γ together with an integral bilinear antisymmetric pairing $\langle\cdot,\cdot\rangle$. Additionally, we will need:

1. A certain charge "lattice" consisting of:
   a) The charge lattice Γ that we already have. We denote elements by $\gamma \in \Gamma$.
   b) A set of Γ-torsors Γ_i labeled by $i \in \mathcal{V}$ whose elements are denoted by $\gamma_i \in \Gamma_i$. These are simultaneously left- and right- Γ-torsors and we impose $\gamma + \gamma_i = \gamma_i + \gamma$.
   c) Γ-torsors
   $$\Gamma_{ij} := \Gamma_i - \Gamma_j \tag{5.1.9}$$
   meaning that its elements are formal differences $\gamma_{ij} := \gamma_i - \gamma_j$ modulo the identification $(\gamma_i + \gamma) - (\gamma_j + \gamma) = \gamma_i - \gamma_j$. This defines a Γ-torsor with an operation $\Gamma_{ij} \times \Gamma_j \to \Gamma_i$. We will denote this operation by '+', e.g. $\gamma_{ij} + \gamma_j \in \Gamma_i$, but warn that $\gamma_j + \gamma_{ij}$ remains undefined for $i \neq j$. There is a canonical identification of Γ_{ii} with Γ (modulo the notational subtlety that an element γ_{ii} is a difference $\gamma_i - \gamma'_i$ and hence generically non-vanishing). Moreover, any element $\gamma_{ij} \in \Gamma_{ij}$ defines a unique element $(-\gamma_{ij}) \in \Gamma_{ji}$ by demanding $\gamma_{ij} + (-\gamma_{ji}) = 0 \in \Gamma_{ii} = \Gamma$ giving rise to a (non-commutative)

addition operation

$$\Gamma_{ij} \times \Gamma_{jk} \to \Gamma_{ik} \tag{5.1.10}$$

which we will also denote by '+'.

d) These data can be gathered in a single groupoid $\mathbb{V}$ in the following way: Let $\mathcal{V}_o$ denote the pointed set consisting of the set $\mathcal{V}$ to which a formal pointed element o is added, i.e. $\mathcal{V}_o := \mathcal{V} \cup \{o\}$. We define $\Gamma_o := \Gamma$ and $\Gamma_{io} := \Gamma_i$ for $i \in \mathcal{V}_o$. The set of objects of $\mathbb{V}$ is given by $\mathcal{V}_o$, i.e. for each $i \in \mathcal{V}_o$ there is an object $V_i \in \mathbb{V}$. Each morphism space $\mathrm{Mor}(V_i, V_j)$, $i, j \in \mathcal{V}_o$, is a Γ-torsor that is identified with Γ_{ij} with composition defined by (5.1.10). The inverse of a morphism $\gamma_{ij} \in \Gamma_{ij}$ is the element $(-\gamma_{ij}) \in \Gamma_{ji}$ (with the convention that the inverse of an element $\gamma_i \in \Gamma_{io} = \Gamma_i$ is given by $-\gamma_i \in \Gamma_{oi} = -\Gamma_i$). The automorphisms $\mathrm{Mor}(V_i, V_i)$ form a group for each $i \in \mathcal{V}_o$ that is canonically isomorphic to Γ which allows the identity morphism to be defined as the neutral element of Γ.

Moreover, each $\gamma \in \Gamma = \Gamma_{oo}$ can naturally be identified with a $\gamma_{ii} \in \Gamma_{ii}$ $(i \in \mathcal{V})$ such that $\gamma + \gamma_{ij} = \gamma_{ii} + \gamma_{ij}$ and $\gamma_{ji} + \gamma = \gamma_{ji} + \gamma_{ii}$. We will henceforth denote any composable elements simply by γ_{ij}, γ_{jk} $(i, j, k \in \mathcal{V}_o)$ to shorten notation because the addition $\gamma_{ij} + \gamma_{kl}$ is defined only if one of the two elements lies in Γ or if $j = k$.

2. A central charge function $Z : \coprod_{i,j \in \mathcal{V}_o} \Gamma_{ij} \to \mathbb{C}$ respecting the torsor structure. In detail the following should hold

   a) $Z|_\Gamma$ is a linear function $\Gamma = \Gamma_{oo} \to \mathbb{C}$.

   b) Z is affine linear on the torsors Γ_i, $i \in \mathcal{V}$, in the sense that $Z_{\gamma+\gamma_i} = Z_\gamma + Z_{\gamma_i}$.

   c) Z is defined on the torsors Γ_{ij}, $i, j \in \mathcal{V}$, by setting $Z_{\gamma_{ij}} := Z_{\gamma_i} - Z_{\gamma_j}$ for $\gamma_{ij} = \gamma_i - \gamma_j$. This is independent of the choice of representatives γ_i, γ_j. Hence we have

$$Z_{\gamma_{ij}+\gamma_{jk}} = Z_{\gamma_{ij}} + Z_{\gamma_{jk}} \tag{5.1.11}$$

   for all $i, j, k \in \mathcal{V}_0$.

3. *4d degeneracies* $\Omega : \Gamma \to \mathbb{Z}$ and *2d degeneracies* $\mu : \coprod_{i \neq j \in \mathcal{V}} \Gamma_{ij} \to \mathbb{Z}$.

4. *Mixed degeneracies* $\omega : \Gamma \times \coprod_{i,j \in \mathcal{V}_o} \Gamma_{ij} \to \mathbb{Z}$ obeying (for $\gamma \in \Gamma$):

   a) For $\gamma' \in \Gamma$:

$$\omega(\gamma, \gamma') = \Omega(\gamma)\langle \gamma, \gamma' \rangle. \tag{5.1.12}$$

   b) For $i, j \in \mathcal{V}_o$:

$$\omega(\gamma, \gamma_{ij} + \gamma_{jk}) = \omega(\gamma, \gamma_{ij}) + \omega(\gamma, \gamma_{jk}). \tag{5.1.13}$$

   c) A *twisting function*

$$\sigma : \coprod_{i,j,k \in \mathcal{V}_o} \Gamma_{ij} \times \Gamma_{jk} \to \mathbb{Z}/2$$

   subject to a cocycle condition for $i, j, k, l \in \mathcal{V}_o$:

$$\sigma(\gamma_{ij}, \gamma_{jk})\sigma(\gamma_{ij} + \gamma_{jk}, \gamma_{kl}) = \sigma(\gamma_{ij}, \gamma_{jk} + \gamma_{kl})\sigma(\gamma_{jk}, \gamma_{kl}). \tag{5.1.14}$$

   as well as a normalization $\sigma(\gamma, \gamma') = (-1)^{\langle \gamma, \gamma' \rangle}$ for $\gamma, \gamma' \in \Gamma_{oo} = \Gamma$.

Having fixed the data, we continue by introducing formal variables $X_{\gamma_{ij}}$ $(i, j \in \mathcal{V}_o)$ with a twisted multiplication

$$X_{\gamma_{ij}} X_{\gamma_{jk}} = \sigma(\gamma_{ij}, \gamma_{jk}) X_{\gamma_{ij}+\gamma_{jk}} \text{ and} \tag{5.1.15}$$

$$X_{\gamma_{ij}} X_{\gamma_{kl}} = 0 \text{ if } \gamma_{ij} + \gamma_{kl} \text{ is not defined.} \tag{5.1.16}$$

Notice that this is associative due to (5.1.14) and agrees with the 4d multiplication (5.1.1) due to the normalization of σ. Next we define the BPS rays which will play the role of the walls at which the jumps occur:

$$\ell_{\gamma_{ij}} := Z_{\gamma_{ij}} \mathbb{R}_- \text{ for } i, j \in \mathcal{V}_o. \tag{5.1.17}$$

The walls l_γ for $\gamma \in \Gamma = \Gamma_{oo}$ and $\omega(\gamma, \cdot) \neq 0$ are called *BPS $\mathcal{K}$-rays* while the walls $\ell_{\gamma_{ij}}$ for $i, j \in \mathcal{V}$ and $\mu_{\gamma_{ij}} \neq 0$ are called *BPS $\mathcal{S}$-rays.* The factors entering the WCF at jumps along $\mathcal{K}$-walls ℓ_γ are the $\mathcal{K}$-factors

$$\mathcal{K}^\omega_\gamma : X_{\gamma_{ij}} \mapsto (1 - X_\gamma)^{-\omega(\gamma, \gamma_{ij})} X_{\gamma_{ij}} \tag{5.1.18}$$

which reduces to (5.1.2) for $i = j = o$. The $\mathcal{S}$-factors arising for jumps at $\mathcal{S}$-walls $\ell_{\gamma_{ij}}$ $(i, j \neq o)$ are maps

$$\mathcal{S}^\mu_{\gamma_{ij}} : X_{\gamma_{kl}} \mapsto \left(1 - \mu(\gamma_{ij}) X_{\gamma_{ij}}\right) X_{\gamma_{kl}} \left(1 + \mu(\gamma_{ij}) X_{\gamma_{ij}}\right). \tag{5.1.19}$$

These can be compared to the pure 2d action (5.1.6) by setting $l = o$ or $k = o$ which yields

$$\mathcal{S}^\mu_{\gamma_{ij}} : \begin{cases} X_{\gamma_k} \mapsto X_{\gamma_k} - \delta_{jk} \mu(\gamma_{ij}) \sigma(\gamma_{ij}, \gamma_j) X_{\gamma_{ij}+\gamma_j}, \\ X_{-\gamma_l} \mapsto X_{\gamma_l} + \delta_{il} \mu(\gamma_{ij}) \sigma(\gamma_{-i}, \gamma_{ij}) X_{\gamma_{-i}+\gamma_{ij}}. \end{cases} \tag{5.1.20}$$

Let us also define $\mathcal{K}^\omega_{\gamma_{ij}} = 1$ for $(i, j) \neq (o, o)$ and $\mathcal{S}^\mu_{\gamma_{ij}} = 1$ for $\{i, j\} \cap \{o\} \neq \emptyset$. Finally, to an angular sector $\sphericalangle \subset \mathbb{C}$ we associate

$$A(\sphericalangle) := \prod^{\curvearrowleft}_{\gamma_{ij} : \ell_{\gamma_{ij}} \subset \sphericalangle} \mathcal{K}^\omega_{\gamma_{ij}} \mathcal{S}^\mu_{\gamma_{ij}} \tag{5.1.21}$$

where the ordering is again in counter-clockwise direction. Then the 2d-4d-WCF states that "$A(\sphericalangle)$ is constant, as long as no BPS ray crosses the boundary of $\sphericalangle$" ([GMN12]).

We will fill this abstract statement with life in the next section by describing alterations to our previous considerations that give rise to the above mentioned structures. Let us finish this section with an observation concerning the data involved for the study of the WCF, namely the role of the $\gamma_i \in \Gamma_i$ (let for this discussion always $i, j \in \mathcal{V}$). It is possible to study the wall-crossing behaviour of the quantities $\Omega(\gamma)$, $\omega(\gamma, \gamma_{ij})$ and $\mu(\gamma_{ij})$ under variations of the $Z_{\gamma_{ij}}$ without explicitly defining $Z_{\gamma_i}, \omega(\gamma, \gamma_i)$ and Γ_i. This will be the case of interest for us where it is impossible to unambiguously define the latter three quantities while we will still be able to define $\Gamma_{ij}, Z_{\gamma_{ij}}, \mu(\gamma_{ij}), \Omega(\gamma)$ and $\omega(\gamma, \gamma_{ij})$ and to study their wall-crossing properties.

5.2 Surface operators and line operators

We want to start by explaining the modifications for string webs, this is work mostly done in [GMN12]. The key ingredient is the *canonical surface defect* $\mathbb{S}_z$ associated to a point $z \in C$. We have encountered surface defects $\mathbb{S}_{\vec{n}}(X_2, \mathcal{R})$ in the description of the six-dimensional theory $\mathfrak{X}$ (see §4.1) and this is one way in which $\mathbb{S}_z$ can be viewed. More precisely, it is located at

$$\{(*,0,0,*)\} \times \{z\} \subset \mathbb{M}^{1,3} \times C, \tag{5.2.1}$$

i.e. in the x^3-t-plane of Minkowski space and at a fixed point of the surface C. Moreover, $\mathcal{R}$ is the fundamental representation of $\mathfrak{su}(2)$ and $\vec{n}$ can generally vary. A *BPS soliton* is a BPS particle that is bound to the defect and interpolates between distinct vacua (that is between different lifts to the cover $\Sigma \to C$). In terms of the geometrical interpretation via string webs we now allow for a single string to end on the point z (recall that all other strings of the web end at either a branch point or a singular point or a three-point-junction). As before, the BPS condition requires us to look particularly at *finite* such webs which will consequently be called *finite open string webs*.

Denote by N one such finite open string web and by N_Σ the lift of N to Σ. The latter has a boundary given by the lifts $z^{(i)}$ and $z^{(j)}$ of z to Σ where we assumed that the string which ends on z is an ij-string. In case that string is oriented out of z, N_Σ is a 1-chain with boundary

$$\partial N_\Sigma = z^{(j)} - z^{(i)} \tag{5.2.2}$$

and hence the charge $[N_\Sigma]$ is a relative homology class on Σ. The relative homology classes obeying (5.2.2) form a set $\Gamma_{ij}(z,z)$ and we let $\Gamma(z,z) := \cup_{i,j}\Gamma_{i,j}(z,z)$ denote the union of all these.

There is a two-dimensional QFT living on the surface defect $\mathbb{S}_z$ which preserves four of the eight supersymmetries of the ambient four-dimensional theory, more precisely the two dimensional theory enjoys $\mathcal{N} = (2,2)$ supersymmetry ([Gai12]). From the two-dimensional point of view, C is the parameter space of couplings for the 2d twisted superpotential which each yield a finite set $\mathcal{V}_z$ ($z \in C$) of massive vacua. These vacua more precisely determine a fibre bundle over the parameter space C (with fibre $\mathcal{V}_z$), hence producing a surface Σ which coincides with the spectral curve of the 4d theory. A striking consequence is that different ways of obtaining the same four-dimensional theory by twisted compactification from six dimensions can be reinterpreted as different choices of the surface defects of the theories of class $\mathcal{S}$ ([Gai12]).

Next we will exploit one of the main strengths of defects, namely that they can themselves contain defects of positive codimension. For the case at hand, it is possible to have line defects that separate two different two-dimensional theories on the defect (this is an example of a *domain wall* since it has codimension one), or more generally *supersymmetric interfaces* that relate different surface defects $\mathbb{S}_{z_1}$ and $\mathbb{S}_{z_2}$. Such an interface will preserve only two of the remaining four supersymmetries, and which of those is determined by a phase $\zeta = e^{i\vartheta}$ and the homotopy type of a non-self-intersecting path $\wp(z_1, z_2) \subset C$ connecting the points z_1 and z_2.[1] We will denote the supersymmetric interfaces by $L_{\wp,\vartheta}$ and note that they have another very natural interpretation, namely as the remnant of a surface defect of the six-dimensional theory $\mathfrak{X}$. More precisely, note that the six-dimensional theory on $\mathbb{M}^{1,3} \times C$ can have three qualitatively different types of surface defects:

1. The surface defect can live on the product (or more generally a fibre bundle) of a two-dimensional submanifold of Minkowski space and a zero-dimensional submanifold of C. This is the situation in which the canonical surface defect $\mathbb{S}_z$ has

[1]We will henceforth assume that only the homotopy type of $\wp$ is relevant. However, due to some tricky sign issues one should really care about a certain "twisted homotopy type" ([GMN13b]).

evolved.

2. The surface defect can live on the product (or more generally a fibre bundle) of a one-dimensional submanifold of Minkowski space and a one-dimensional submanifold of C.

3. The surface defect can live on the product (or more generally a fibre bundle) of a zero-dimensional submanifold of Minkowski space and a two-dimensional submanifold of C.

The supersymmetric interface $L_{\wp,\vartheta}$ clearly falls into the second category and should more precisely be denoted $L_\vartheta(\mathcal{R}, q, \vec{n}; \wp)$ with a representation $\mathcal{R}$ of $\mathfrak{su}(2)$, a path q in $\mathbb{M}^{1,3}$ and a map $\vec{n} : q \times \wp \to \mathbb{R}^5$. However, we will fix $\mathcal{R}$ to be the fundamental representation as before, $q = \mathbb{R} \times \{\vec{0}\}$ to be the line along the time direction in $\mathbb{M}^{1,3}$ and $\vec{n}$ in such a way that it has a constant angle ϑ with the tangent vector to $\wp$ when it is transported along $\wp$ (the latter is not entirely correct and depends on the choice of a subspace $\mathbb{R}^2 \subset \mathbb{R}^5$, see [GMN13a] for details).

There is one more tool which is absolutely crucial for what is to come, namely *2d-4d framed BPS states*. We have just stated that a supersymmetric interface $L_{\wp,\vartheta}$ is defined by a choice of two points $z_1, z_2 \in C$ together with a path $\wp \subset C$ connecting them and a phase $\zeta = e^{i\vartheta}$. In the four-dimensional picture we obtain an interpretation of the supersymmetric interface $L_{\wp,\vartheta}$ as the interface located at $\mathbb{R} \times \{\vec{0}\}$ separating the two half-planes $\mathbb{R} \times \{0, 0\} \times \mathbb{R}_{>0}$ and $\mathbb{R} \times \{0, 0\} \times \mathbb{R}_{<0}$ on which the surface defects $\mathbb{S}_{z_1}$ and $\mathbb{S}_{z_2}$ are inserted, respectively. This system contains 2 supercharges defined by ϑ and we define $\mathcal{H}^1_{L_{\wp,\vartheta}}$ to be the one-particle Hilbert space of this combined system.

Definition 5.2.1. A *2d-4d framed BPS state* is a state in $\mathcal{H}^1_{L_{\wp,\vartheta}}$ which preserves both of the present supercharges.

This is an extension of the pure 4d framed BPS states considered in [GMN13b] which was introduced in [GMN12]. Under certain genericity assumptions they can be viewed as special vacuum states of the interface ([GMN13b]). There is an alternative way to view these states: Denote by $\mathcal{H}^1_L$ the Hilbert space of 1-particle states in the presence of the line defect $L := L_{\wp,\vartheta}$. In analogy to (3.3.3) this space is graded by a Γ-torsor Γ_L, i.e.

$$\mathcal{H}^1_L = \bigoplus_{\gamma \in \Gamma_L} \mathcal{H}^1_{L,\gamma} \tag{5.2.3}$$

and on these subspaces there is the "twisted" BPS bound

$$m \geq -\mathrm{Re}(Z_\gamma/\zeta). \tag{5.2.4}$$

2d-4d framed BPS of charge γ are states in $\mathcal{H}^1_{L,\gamma}$ saturating this bound.

In terms of charges associated to paths on C and their lifts to Σ, we must now work (almost) entirely on Σ because the lifts describe different vacua of the 2d-theory. More precisely, rather than to look for a path on Σ ending on lifts $z^{(i)}$ and $z^{(j)}$ of the *same* base point $z \in C$, we require the ends to be the union of a lift $z_1^{(i)}$ of z_1 and a lift $z_2^{(j)}$ of z_2. In analogy to the former case, these paths define relative cohomology classes which we gather in a set $\Gamma_{ij}(z_1, z_2)$, or more generally $\Gamma(z_1, z_2) := \cup_{i,j}\Gamma_{ij}(z_1, z_2)$.

We are now in the position to define the data described in the previous section for the situation at hand. Let us fix points $u \in \mathcal{B}$ and $z \in C$, then the data is given as follows:

1. The charge "lattice":

   a) The lattice Γ remains the same as in the 4d case, i.e. is the lattice Γ_u over $u \in \mathcal{B}^*$ defined in (4.4.5).

   b) The torsor Γ_i has no obvious interpretation and we will neglect it henceforth. Physically speaking, it parametrizes the values of the IR superpotential $\mathcal{W}$ in the vacuum V_i but these carry certain anomalies (see [GMN12] §3.4.1 for details).

   c) $\Gamma_{ij} := \Gamma_{ij}(z,z)$ parametrizes 1-chains on Σ with boundary $z^{(i)} - z^{(j)}$ modulo boundaries. Addition is via concatenation of paths and the inverse is given by the same path with opposite orientation. This defines a Γ-torsor because two relative homology paths differ by a loop on Σ. There is a canonical isomorphism $\Gamma_{ii}(z,z) \xrightarrow{\sim} \Gamma$ but this breaks down for the more general case of two different endpoints. [GMN12] explores circumstances under which this lack of an isomorphism can be compensated by monodromies of the flavour charges as in (4.4.8).

2. The central charge functions Z are defined in analogy to (4.4.7) as

$$Z_a := \frac{1}{\pi} \oint_a \lambda \tag{5.2.5}$$

   for $a = \gamma \in \Gamma$ or $a = \gamma_{ij} \in \Gamma_{ij}$. It is clear that $Z_a + Z_b = Z_{a+b}$ for composable paths a, b.

3. The 4d index Ω remains the same as in (3.3.4) and is defined as a certain trace over the Hilbert space of 1-particle BPS states. Similarly one can define the index $\mu : \Gamma_{ij} \to \mathbb{Z}$ as the trace

$$\mu(a) := \mathrm{tr}_{\mathcal{H}^{1,\mathrm{BPS}}_{\mathbb{S}_z,a}} F e^{i\pi F} \tag{5.2.6}$$

   as the trace over the 1-particle Hilbert space of BPS states living on the surface defect $\mathbb{S}_z$ and carrying charge $a \in \Gamma(z,z)$. We will not take this as a working definition but rather as a remark that this index can be defined in a physical context. Instead, we will give a more indirect characterization of μ through theorem 5.3.5. However, note that F denotes a fermion number operator, i.e. a charge of a factor $\mathfrak{u}(1)_V$ of the $2d\ \mathcal{N} = (2,2)$ supersymmetry algebra preserved by $L_{\wp,\vartheta}$. This definition is well-known for two-dimensional theories and goes back to [CFIV92] but is somewhat more subtle in the 2d-4d context giving rise to a sign ambiguity which is addressed in [GMN13b].

4. There is no simple definition of the 2d-4d-indices $\omega(\gamma, \gamma_{ij})$ in terms of a trace-formula over a Hilbert space that we can explain in the scope of this thesis. In the context of supergravity there are *halo states* which obey a version of the KSWCF (see [ADJM12]) which induces the 2d-4d wall-crossing considered here ([GMN12]). However, ω can be computed via

$$\omega(\gamma, \gamma_{ij}) := \Omega(u, \gamma)\langle \gamma, \gamma_{ij} \rangle \tag{5.2.7}$$

   where $\langle \cdot, \cdot \rangle$ denotes the pairing between the homology group $H_1(\Sigma - \pi^{-1}(z); \mathbb{Z})$ and the relative homology of paths ending on preimages of z. The twisting function σ is given by $\sigma(a,b) = (-1)^{\langle a,b \rangle}$.

Remark 5.2.2. Recall that in the pure 4d context we have explained that the KSWCF can be interpreted as a smoothness condition for a Hyperkähler metric on the moduli space of a three-dimensional theory obtained by compactifying over a circle. A similar statement holds for the 2d-4d WCF: Compactifying the four-dimensional bulk theory together with the defect yields a three-dimensional theory with a supersymmetric line operator (when one of the dimensions of the surface defect $\mathbb{S}_z$ is wrapped on the circle). In more mathematical terms, one obtains a hyperholomorphic connection on a vector bundle over a Hyperkähler manifold.[2] Demanding smoothness of the metric and the connection is then equivalent to the 2d-4d WCF ([GMN12]).

5.3 The Formal Parallel Transport Theorem

We are interested in one more index, namely the *framed BPS index* $\underline{\overline{\Omega}}(L_{\wp,\vartheta}, a)$ associated to a pair $(z_1, z_2) \in C \times C$ connected by a path $\wp$, a phase ϑ and the associated supersymmetric interface $L_{\wp,\vartheta}$ as well as a charge $a \in \Gamma(z_1, z_2)$. This index can be defined similar to (5.2.6) via a trace

$$\underline{\overline{\Omega}}(L_{\wp,\vartheta}, a) := \mathrm{tr}_{\mathcal{H}^{1,\mathrm{BPS}}_{L_{\wp,\vartheta},a}} e^{i\pi F} \tag{5.3.1}$$

where again F is the choice of a $\mathfrak{u}(1)_V$ charge generator. As before, we will not take this as the definition but give a more indirect approach through the Formal Parallel Transport Theorem (theorem 5.3.5). For the moment, assume the indices μ and $\underline{\overline{\Omega}}$ were defined, we will use them for the definition of the *formal parallel transport* and the *Spectral Network*. The statement of the Formal Parallel Transport Theorem will be that the indices μ and $\underline{\overline{\Omega}}$ are determined by certain properties of the formal parallel transport and spectral networks, hence making a definition via a trace redundant. First we need the notion of the homology 1-groupoid:

Definition 5.3.1. (Homology 1-groupoid)
The *homology 1-groupoid* $H_{\leq 1}(X)$ (also known as the *homology path algebra*) of a topological space X is the groupoid defined as follows:

- Objects are points of X.
- $H_{\leq 1}(x, y)$ denotes the morphism space between objects x, y and is defined as the quotient
$$H_{\leq 1}(x, y) := C_1(x, y)/\sim \tag{5.3.2}$$
where $C_1(x, y)$ is the set of 1-chains $\mathfrak{c}$ with $\partial\mathfrak{c} = y - x$ and the equivalence relation states that $\mathfrak{c}_1 \sim \mathfrak{c}_2$ if $\mathfrak{c}_1 - \mathfrak{c}_2$ is a 1-boundary. Hence, $H_{\leq 1}(x, y)$ is an affine subspace of the relative homology $H_1(X, \{x, y\}; \mathbb{Z})$.
- Composition of morphisms is induced by addition of chains with neutral element the trivial path (a point) and inversion by reversion of orientation.

Remark 5.3.2. The morphism space $H_{\leq 1}(x, y)$ is a torsor for $H_1(X; \mathbb{Z})$.

Remark 5.3.3. There is a natural functor $\pi_{\leq 1}X \to H_{\leq 1}(X)$ with domain the fundamental 1-groupoid defined in example 2.2.13. In particular, the homotopy class of a smooth path (with fixed endpoints) determines a morphism in the homology 1-groupoid.

[2] Recall that a hyperholomorphic connection is defined to be a connection whose curvature is of type (1, 1) in all the complex structures of the Hyperkähler manifold. This can be viewed as a generalization of the instanton equations for four-dimensional Hyperkähler manifolds.

Denote to a path a on C the corresponding morphism of the homology 1-groupoid $H_{\leq 1}(C)$ by X_a. In particular, their product is given by

$$X_a X_b = \begin{cases} X_{a \circ b} & \text{end}(a) = \text{ beg}(b), \\ 0 & \text{else}, \end{cases} \tag{5.3.3}$$

where $a \circ b$ denotes the homology class of the concatenated path of any two representatives. We define the *formal parallel transport*

$$F(\wp, \vartheta) := \sum_{a \in \Gamma(z_1, z_2)} \underline{\overline{\Omega}}(L_{\wp,\vartheta}, a) X_a. \tag{5.3.4}$$

This definition does not depend on the choice of a representative a ([GMN13b]). We are interested in ways to compute the formal parallel transport and for this purpose we introduce spectral networks:

Definition 5.3.4. (Spectral Networks)
Fix an angle ϑ. The spectral network $\mathcal{W}_\vartheta$ is defined as the set

$$\mathcal{W}_\vartheta := \left\{ z \in C \,\middle|\, \exists\, a \in \Gamma(z,z) : Z_a / e^{i\vartheta} \in \mathbb{R}_- \text{ and } \mu(a) \neq 0 \right\}. \tag{5.3.5}$$

$\mathcal{W}_\vartheta$ is a network of codimension-1 segments called $\mathcal{S}$*-walls* on C. For generic ϑ and generic $z \in \mathcal{W}_\vartheta$ there is a unique charge $a \in \Gamma(z,z)$ satisfying the above conditions. We say that z supports the charge a and that z sits on an ij-wall for $a \in \Gamma_{ij}(z,z)$. From now on we will always assume that ϑ is generic (unless stated otherwise but we will denote critical values by ϑ_c).

Note that while μ does not enter the definition of the formal parallel transport $F(\wp, \vartheta)$, the definition of the spectral network $\mathcal{W}_\vartheta$ does not depend on a definition of the framed BPS index $\underline{\overline{\Omega}}(L_{\wp,\vartheta}, a)$. However, the strong interplay between the indices projects to these definitions in a fascinating way, captured by the *Formal Parallel Transport Theorem* due to Gaiotto, Moore and Neitzke:

Theorem 5.3.5. (Formal Parallel Transport Theorem)
There is a unique set of BPS indices

1. $\mu(a)$ *for* $a \in \Gamma(z,z), z \in C$ *and*
2. $\underline{\overline{\Omega}}(L_{\wp,\vartheta}, a)$ *for* $z_1, z_2 \in C \backslash \mathcal{W}_\vartheta$*,* $a \in \Gamma(z_1, z_2)$*,* $\wp$ *a path from* z_1 *to* z_2

determining $\mathcal{W}_\vartheta$ *as in* (5.3.5) *and* $F(\wp, \vartheta)$ *as in* (5.3.4) *such that the following conditions hold:*

1. (Homotopy invariance) $F(\wp, \vartheta) = F(\wp', \vartheta)$ *for homotopic paths* $\wp, \wp'$ *with fixed endpoints in* C*.*
2. (Homomorphism property) $F(\wp, \vartheta) F(\wp', \vartheta) = F(\wp \circ \wp', \vartheta)$ *for paths in* C *with end*$(\wp)$ = *beg*$(\wp')$ *whose endpoints are not on* $\mathcal{W}_\vartheta$*. Here* $\wp \circ \wp'$ *denotes their concatenation.*
3. (Local triviality) *If* $\wp \cup \mathcal{W}_\vartheta = \emptyset$ *then*

$$F(\wp, \vartheta) = \sum_{i=1}^{K} X_{\wp^{(i)}} =: D(\wp) \tag{5.3.6}$$

where $\wp^{(i)}$ *denotes the distinct lifts to the* $K:1$*-cover* $\Sigma \to C$

4. (Detour rule) *Let $\wp$ a small path with $\wp \cup \mathcal{W}_\vartheta = z$ such that z is generic and supports the charge $a \in \Gamma_{ij}(z,z)$. Denote the segment of $\wp$ oriented into (out of) the network by $\wp_+$ ($\wp_-$). Then*

$$\begin{aligned} F(\wp,\vartheta) &= D(\wp_+)\,(1+\mu(a)X_a)\,D(\wp_-) \\ &= D(\wp) + D(\wp_+)\,(\mu(a)X_a)\,D(\wp_-) \\ &= D(\wp) + X_{\wp_+^{(i)}}\,(\mu(a)X_a)\,X_{\wp_-^{(j)}}. \end{aligned} \tag{5.3.7}$$

Remark 5.3.6. Concerning the detour rule (Property 4) there are a couple remarks in order:

1. By a *small* path we mean a path that intersects the spectral network only once. Due to the homomorphism property an arbitrary path can be split into pieces that are either disjoint to the Spectral Network (and where F can hence be computed by property 3) or intersect the Spectral Network in only one point and do not stretch very far from it (and can hence be computed by property 4).

2. Given a small path, one can always find a homotopic path such that its intersection with $\mathcal{W}_\vartheta$ is a generic point of the Spectral Network, and by property 1 the two paths yield the same contribution to F.

3. The three expressions on the right side of (5.3.7) are trivially equivalent due to the product structure on the homology 1-groupoid given by (5.3.3) but they are all convenient in different situations. For technical reasons, the paths $\wp^\pm$ should be slightly deformed to be tangent to $\mathcal{W}_\vartheta$ at z so that they are indeed composable smoothly with the paths in $\Gamma_{ij}(z,z)$ but we have omitted that in our notation.

4. The detour rule can be read as a WCF for the framed indices $\overline{\underline{\Omega}}(L_{\wp,\vartheta},a)$ in the following way: Associate a supersymmetric interface $L_{\wp_x,\vartheta}$ to the path $\wp_x = \wp\big|_{[z_1,x]}$ where $\wp$ is the path connecting z_1 and z_2 in C and x varies over the points of this path. As x crosses the $\mathcal{S}$-wall, $F(\wp_x,\vartheta)$ jumps by multiplication with a factor $(1+\mu(a)X_a)$ which means that the supersymmetric interface can absorb or emit a BPS soliton. It is hence an example for the CVWCF for two-dimensional theories, which is what one would expect for solitons. Since the CVWCF is expressed in terms of $\mathcal{S}$-factors as in (5.1.6), the walls at which these jumps occur are called $\mathcal{S}$-walls.

Remark 5.3.7. This theorem holds for arbitrary values of ϑ without any genericity assumption. However, we need a little rephrasing for the detour rule (Property 4) for critical values of ϑ. The essential phenomenon occuring is that each element z of an $\mathcal{S}$-wall ℓ supports two charges $a \in \Gamma_{ij}(z,z)$ and $b \in \Gamma_{ji}(z,z)$. In this case, ℓ is called a *two-way street* and the value ϑ_c at which the jump occurs is a *$\mathcal{K}$-wall*. We will take a closer look at the behaviour of Spectral Networks for critical values of ϑ in section 5.5.

The proof of theorem 5.3.5 has been carried out in [GMN13b] and we do not wish to copy it here. However, we will sketch it through the next two sections where we take a closer look at properties of Spectral Networks first for generic values of θ and then at special values ϑ_c.

5.4 Spectral Networks at generic ϑ

Let for this section ϑ always be generic. Firstly, note that the lattices $\Gamma(z,z)$ form a local system over C which allows for a parallel transport of homology classes $\Gamma(z,z) \ni a_z \to a_{z'} \in \Gamma(z',z')$ along a path connecting z and z' via a lift to the fibers. In particular, if there is a soliton $a \in \Gamma_{ij}(z,z)$ for z a point on an $\mathcal{S}$-wall ℓ, the index $\mu(a)$ is unchanged while z moves along ℓ but it might jump when two (or more) $\mathcal{S}$-walls collide. We consequently say that ℓ supports the charge $a \in \Gamma_{ij}$ meaning that it does so for all points of ℓ. If that is the case, then $e^{-i\vartheta} Z_a$ is real along ℓ and with

$$\mathrm{d}Z_a = \frac{1}{\pi}\left(\lambda^{(i)} - \lambda^{(j)}\right) \tag{5.4.1}$$

we see that ℓ is a WKB curve of type ij and inclination θ in the sense of section 4.5. Consequently, in analogy to the derivation of (4.5.10) we see that the mass of a soliton located on an ij-wall of the Spectral Network near a branch point $\mathfrak{b}$ has mass

$$M(z) \sim |z|^{3/2} \tag{5.4.2}$$

for a local coordinate z that vanishes at $\mathfrak{b}$.

We now introduce the *mass filtration* $\mathcal{W}_\vartheta[\Lambda]$ ($\Lambda \in \mathbb{R}_+$) by truncating all the $\mathcal{S}$-walls: it contains only that portion of every wall ℓ of $\mathcal{W}_\vartheta$ supporting the soliton of charge a for which $|Z_a| < \Lambda$. This implies that

$$\mathcal{W}_\vartheta[\Lambda] \subset \mathcal{W}_\vartheta[\Lambda'] \text{ for } \Lambda < \Lambda' \tag{5.4.3}$$

and

$$\lim_{\Lambda\to\infty} \mathcal{W}_\vartheta[\Lambda] = \mathcal{W}_\vartheta. \tag{5.4.4}$$

as topological spaces. In the same spirit we truncate μ via

$$\mu[\Lambda](a) := \begin{cases} \mu(a) & \text{for } |Z_a| < \Lambda, \\ 0 & \text{else.} \end{cases} \tag{5.4.5}$$

This gives the following roadmap to understand Spectral Networks better: Start with small values of Λ for which $\mathcal{W}_\vartheta[\Lambda]$ can be computed easily, then evolve the Spectral Network by letting $\Lambda \to \infty$.

Recall that the solitons are BPS particles, hence their mass must agree with the absolute value of their charge. However, their mass depends on the distance to the closest branch point according to (5.4.2), hence for small Λ only solitons close to a branch point contribute. The Spectral Network must then consist of the branch points together with three short WKB curves emitted from each branch point. More precisely, near an (ij)-branch point $\mathfrak{b}$ we can choose a local orientation on the complement of the corresponding (ij)-branch cut (this orientation cannot be consistent globally since monodromy around the branch point exchanges the two sheets) such as depictured in figure 5.1. Note that in particular away from the branch cut the orientation of the $\mathcal{S}$-walls alternates in accordance with the fact that there is a whole WKB foliation of which the Spectral Network contains only the critical WKB curves.

So how do the indices μ behave. It is shown in [GMN13b] that $\mu(a) = 1$ for any $a \in \Gamma(z,z)$ and z a point of an $\mathcal{S}$-wall near the branch point $\mathfrak{b}$. This is essentially a consequence of the homotopy invariance by looking at the jumping behaviour determined

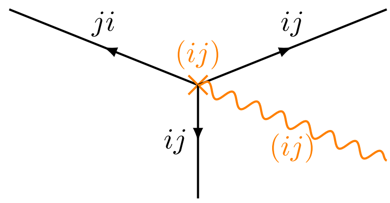


FIGURE 5.1: A generic Spectral Network near a branch point of type (ij) (orange cross): The black lines picture the three $\mathcal{S}$-walls emitted from the branch point. Crossing the branch cut (orange "snake" line) exchanges the sheets i and j of the covering.

by the detour rule for a path winding around the branch point. However, the correct analysis contains some pesky sign issues and we are hence not going to repeat it here.

Know that the initial conditions are fixed, we can start to grow the Spectral Network by continuously increasing Λ. Consequently, the WKB curves start to get longer without changing the analysis above until they start to intersect transversly. We call points in which $\mathcal{S}$-walls intersect *joints*. The intersecting walls can be of three different types:

1. The two $\mathcal{S}$-walls could carry disjoint labels ij and kl. In that case, they would just pass through one another without any changes in the indices μ.

2. An ij- and a jk-wall could meet transversely in the joint z. Consequently, the joint supports two charges $a \in \Gamma_{ij}(z, z)$ and $b \in \Gamma_{jk}(z, z)$ which can be composed to a charge $c = a + b \in \Gamma_{ik}(z, z)$. Hence one might expect the appearance of ik-walls which is indeed the case. For generic μ the local picture is as in figure 5.2. We will describe this in more detail below.

3. One could expect that an ij- and a ji-wall intersect transversely but this is impossible because the two walls must solve the same differential equation and can hence not meet transversely.

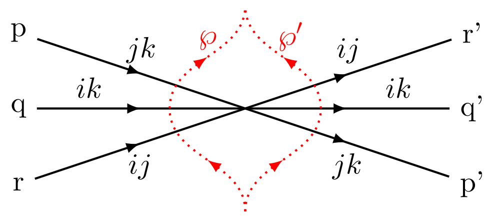


FIGURE 5.2: Around a generic collision of an ij-wall with a jk-wall, there are ik-walls appearing. Homotopy invariance of the Formal Parallel Transport F demands that the combined wall-crossings along $\wp$ and $\wp'$ agree.

The generic case of the collision of an ij-wall and a jk-wall is depicted in figure 5.2. Homotopy invariance demands that

$$F(\wp, \vartheta) = F(\wp', \vartheta) \tag{5.4.6}$$

which translates in terms of the detour rule (5.3.7) into the condition

$$\begin{aligned} &D(\wp_+)\,(1+\mu(a,r)X_a)\,(1+\mu(c,q)X_c)\,(1+\mu(b,p)X_b)\,D(\wp_-) \\ =&D(\wp_+)\,(1+\mu(b,p')X_b)\,(1+\mu(c,q')X_c)\,(1+\mu(a,r')X_a)\,D(\wp_-). \end{aligned} \tag{5.4.7}$$

Using the relations of the homology 1-groupoid (5.3.3) one hence obtains the relations

$$\begin{aligned} \mu(a,r') &= \mu(a,r), \\ \mu(b,p') &= \mu(b,p), \\ \mu(c,q') &= \mu(c,q) \pm \mu(a,p)\mu(b,r), \end{aligned} \tag{5.4.8}$$

where the sign issue that we have been carrying along is reflected in the last equation. This sign can be determined unambiguously, which is done in [GMN13b], but we will not dwell on it. The important fact is that the number of solitons carrying charge c changes upon crossing the joint which is caused by the formation or decay of bound states between solitons of charges a and b.

In particular, it is possible that there is no incoming soliton of charge c (that is, $\mu(c,q)=0$) or no outgoing soliton of charge c (i.e. $\mu(c,q')=0$). Correspondingly, one would have only two $\mathcal{S}$-walls colliding and giving birth to a new $\mathcal{S}$-wall of type ik, or an ik-wall being annihilated in the collision process. These cases are depicted in figure 5.3. In either case, the value of μ for the charge c is determined by the values for charges of a and b as a simplification of the third equation in (5.4.8).

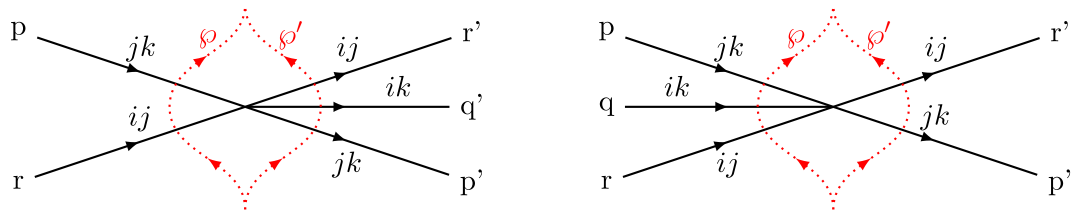


FIGURE 5.3: Two walls of type ij resp. jk can collide and produce a new wall of type ik (left). By reversal of orientation one sees that in the same spirit three walls can collide with one of them dying at the collision point.

Let us recall what we have seen so far: The Spectral Network $\mathcal{W}_\vartheta$ comes with a natural filtration $\mathcal{W}_\vartheta[\Lambda]$ for $\Lambda \in \mathbb{R}_+$ that asymptotes the Spectral Network for $\Lambda \to \infty$. For small values of Λ, only simpletons contribute to the Spectral Networks, i.e. $\mathcal{W}_\vartheta[\Lambda]$ consists of the branch points with three short WKB curves coming out of each branch point. As we increase Λ, these curves start growing according to the differential equation (4.5.6) with their length determined by the cutoff Λ. Keeping on with this procedure, the $\mathcal{S}$-walls can start meeting transversally, potentially giving birth to new $\mathcal{S}$-walls or ending them as in figure 5.3. Eventually, the $\mathcal{S}$-walls approach the basin of attraction around the singular points of the surface C where they get trapped and fall into the singularity as logarithmic spirals which we have described in section 4.5.

We have not yet explained what happens if walls meet *not-transversally*. For example, an ij-wall and a ji-wall can run into each other head-on. This gives rise to $\mathcal{K}$-walls and only happens at non-generic values ϑ_c as we will see in the next section.

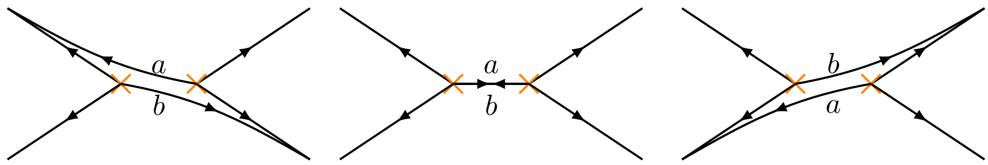
5.5 Spectral Networks at critical ϑ

So far we have looked at Spectral Networks $\mathcal{W}_\vartheta$ and the functions $F(\wp, \vartheta)$ for fixed and generic θ. In this section we will summarize what happens as ϑ is varied. Starting from a generic value of ϑ and hence the situation described in the previous section, small enough changes of ϑ leave the topology of the spectral network invariant and also do not lead to any changes of $F(\wp, \vartheta)$. In other words, F is piecewise constant as a function of ϑ but has a jumping behaviour for two very different reasons:

1. As ϑ is varied, the Spectral Network can run over one of the two endpoints of $\wp$ at a critical value ϑ_c. This is equivalent to saying that upon extending the path $\wp$, it crosses $\mathcal{W}_{\vartheta_c}$. Hence the change is governed by the Detour Rule (5.3.7). In particular, the function F jumps through the multiplication of an $\mathcal{S}$-factor and the Spectral Network does not change its topology.

2. The second and more interesting possibility is that the Spectral Network does change its topology. This leads to the notion of a $\mathcal{K}$-wall (corresponding to the fact that the change in F is governed by a four-dimensional $\mathcal{K}$-factor) and will be the subject of this section.

Figure 5.4 shows the easiest example of how a $\mathcal{K}$-wall can arise. The collection of ϑ_c where the topology of the Spectral Network jumps are called $\mathcal{K}$-walls. The different ways in which $\mathcal{K}$-walls can arise are described in [GMN13b] but the basic principle remains the same as in the case depictured in figure 5.4.



(A) $\vartheta < \vartheta_c$ (B) $\vartheta = \vartheta_c$ (C) $\vartheta > \vartheta_c$

FIGURE 5.4: Two distinct $\mathcal{S}$-walls supporting the charges a and b, respectively, merge into a single wall supporting both of the charges as $\vartheta \to \vartheta_c$, giving rise to a two-way street. Upon further increasing ϑ one obtains a generic Spectral Network of different topology.

To explore the phenomenon of $\mathcal{K}$-walls it is convenient to denote the two limits by $\vartheta_c^\pm := \vartheta_c \pm \epsilon$ for an infinitesimal $\epsilon \searrow 0$. The claim is that there is a map $\mathcal{K}$ acting uniformly on the formal variables X_a such that

$$F(\wp, \vartheta_c^+) = \mathcal{K}\left(F(\wp, \vartheta_c^-)\right). \tag{5.5.1}$$

To formulate how $\mathcal{K}$ acts on the homology 1-groupoid, we start by letting $\Gamma_c \subset \Gamma$ be the subset

$$\Gamma_c := \left\{\gamma \in \Gamma \,\middle|\, e^{-i\vartheta_c} Z_\gamma \in \mathbb{R}_-\right\}. \tag{5.5.2}$$

We make the genericity assumption that there is a single γ_c generating Γ_c. This holds if $u \in \mathcal{B}^*$ but breaks down at the wall of marginal stability. Next recall that in section 4.5 we have defined the lift p_Σ of an ij-string p from C to the cover Σ as the union of the lift $p^{(i)}$ to the i-th sheet and of the lift $-p^{(j)}$ of the lift to the j-th lift with reversed orientation. Then $\mathcal{K}$ is the following morphism:

$$\mathcal{K}(X_a) = \prod_{\gamma \in \Gamma_c} (1 - X_\gamma)^{-\omega(\gamma, a)} X_a \tag{5.5.3}$$

which is a $\mathcal{K}$-factor of the form (5.1.18), hence explaining the name $\mathcal{K}$-wall. Indeed, the appearance of $\mathcal{K}$-walls must then be directly related to 4d BPS states because the jumping behaviour of their indices is governed by the $\mathcal{K}$-factors. In [GMN13b] it is shown that for the saddle connection depicted in figure 5.4 for $\vartheta = \vartheta_c$ one has $\Omega(\gamma) = 1$ where $\gamma = a + b \in \Gamma$ (after forgetting the basepoint). Hence, a saddle connection represents a BPS hypermultiplet (recall (3.2.4) and (3.3.5))! In a similar matter one can find that vectormultiplets are represented by closed loops ([GMN13b], §7.2).

Remark 5.5.1. With the reappearance of the index Ω at this point it seems like a natural question whether Ω (like μ and ω) can be obtained geometrically and without the definition via a trace formula. That is indeed the case and is crucially connected to the sign issue we have not resolved in this discussion of Spectral Networks. A precise account is given in [GMN13b].

A proof that the Formal Parallel Transport F indeed jumps as described by (5.5.1) and (5.5.3) is given in [GMN13b]. In fact, this jumping behaviour is again strong enough to determine the indices μ exactly which finishes our sketch of a proof for theorem 5.3.5.

We have hence seen the close relationship that the 2d-4d WCF and the Formal Parallel Transport Theorem enjoy: The detour rule for generic resp. critical ϑ corresponds to the jumping by an $\mathcal{S}$- resp. a $\mathcal{K}$-factor. In more physical terms, the former describe how 2d-solitons are formed (or annihilated), while the latter describe the jumping behaviour for 4d BPS states.

Chapter 6

Conclusion and outlook

The aim of this thesis was to bridge a gap between the mathematics and physics around Spectral Network. Even though both the usefulness and the definition of these objects stand without a doubt in the mathematics community, their motivation from a physical perspective remains difficult to grasp for pure mathematicians.

Our starting point were certain quantum field theories which can physically be classified as four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry and which are associated to a (semi-simple) Lie group. To understand these theories better, it is crucial to get a grasp on the fundamental pieces occuring within them, which we have explained to be the so-called BPS states. Despite the fact that these states have been the fruitful subject of study in a few exceptional cases, they have remained outside of reach for a long time. We explored two ways of dermining their spectrum in two limited cases, one of them being the isolated theory associated to the group SU(2). Gaiotto, Moore and Neitzke introduced the powerful tool of Spectral Networks that not only extended the previous considerations by allowing for more general groups, but that also provide a more elegant and efficient way in which the number of BPS states can be computed.

Even though this thesis has its share of pages, it also contains more gaps than the writer would like to admit. Some of these could be filled with the proper time at hand, others go beyond the scope of this work, as for example:

1. Higher categories as introduced in section 2.2 remain infeasible objects to define in a straightforward fashion. The common approach is hence to describe them indirectly through *model categories*. However, this procedure yields a variety of different definitions whose interplay has barely been started to disentangle.
2. The problem of extending (non-topological) Functorial Quantum Field Theories as introduced in section 2.3 to the spheres of higher category theory remains largely untouched. Such a framework would be particularly interesting for work relying heavily on compactification and defects, such as the present thesis.
3. The IR limit of compactification stands without a proper definition in the scope of FQFTs. One might want to introduce a topology on the space of field theory functors but this seems outside of reach while the higher functors remain without a proper definition.
4. The close relationship Donaldson-Thomas invariants enjoy with the BPS indices defined in this thesis have not been explained. This should be rather unsurprising from the way the BPS indices have entered the Wall-Crossing Formula of Kontsevich and Soibelman.
5. It would be interesting to describe how the considerations made here for gauge theories extend to the scope of supergravity. We have seen that there is currently

no understanding of the 2d-4d indices ω outside of gravity, but that their is an explanation around *halo* states. The KSWCF still applies to supergravity which is at the heart of [DM11].

6. There remains much unknown about the mysterious Theory $\mathfrak{X}$, and more generally about M-Theory. Their study is a long-lasting and wide-spreading project that draws on progress in many major fields of mathematics and physics.

7. We want to emphasize that one should take into account irregular singular points on C where the differentials have higher order poles. Their impact is described in more detail in [GMN13c].

8. It would be desirable to extend the results described here to the theories of class $\mathcal{S}$ that arise by compactifying over a Riemann surface *without* punctures. There are no basins of attraction for the WKB curves, which can hence generically not end. These theories behave qualitatively very different and have not been much explored to the best knowledge of the author.

9. Similarly, we have focused our attention to theories with gauge group A_{K-1}. One could naturally thrive to explain the situation for more general groups, the results should be somewhat similar at least for the simple, simply laced groups of type D and E.

10. Spectral Networks provide useful tools for (higher) Teichmüller theory, a fact that is explored in [HN13] and [GMN14].

11. There is a natural relation of Spectral Networks to the moduli of flat connections which is adressed in some detail in [GMN13b].

Appendix A

Some category theory

A.1 Symmetric monoidal categories

Symmetric monoidal categories are very well studied objects in mathematics, all definitions spelled out here have been carried out countless times before. A good reference is as always [ML71]. We will not define what a category, a functor and a natural transformation are but start with the following definition:

Definition A.1.1. (Monoidal category)
A *monoidal category* $\mathcal{M}$ consists of a category $\mathcal{M}_0$ together with (for arbitrary $a, b, c \in \mathrm{ob}(\mathcal{M}_0)$):

- a functor $\otimes : \mathcal{M}_0 \times \mathcal{M}_0 \to \mathcal{M}_0$ (called tensor product),
- an object $I \in \mathrm{ob}(\mathcal{M}_0)$ (called tensor unit),
- a natural isomorphism $\alpha_{a,b,c} : (a \otimes b) \otimes c \to a \otimes (b \otimes c)$ (called associator),
- a natural isomorphism $l_a : I \otimes a \to a$ (called left unitor),
- a natural isomorphism $r_a : a \otimes I \to a$ (called right unitor),

subject to two coherence relations, expressed by making the following diagrams commute:

$$\begin{array}{ccccc}
((a \otimes b) \otimes c) \otimes d & & \xrightarrow{\qquad\qquad \alpha_{a\otimes b,c,d} \qquad\qquad} & & (a \otimes b) \otimes (c \otimes d) \\
\downarrow{\scriptstyle \alpha_{a,b,c} \otimes \mathrm{id}_d} & & & & \downarrow{\scriptstyle \alpha_{a,b,c\otimes d}} \\
(a \otimes (b \otimes c)) \otimes d & \xrightarrow{\alpha_{a,b\otimes c,d}} & a \otimes ((b \otimes c) \otimes d) & \xrightarrow{\mathrm{id}_a \otimes \alpha_{b,c,d}} & a \otimes (b \otimes (c \otimes d))
\end{array} , \tag{A.1.1}$$

called the *pentagon identity* (for arbitrary $a, b, c, d \in \mathrm{ob}(\mathcal{M}_0)$) as well as

$$\begin{array}{ccc}
(a \otimes I) \otimes b & \xrightarrow{\qquad \alpha_{a,I,b} \qquad} & a \otimes (I \otimes b) \\
& {\scriptstyle r_a \otimes \mathrm{id}_b} \searrow \qquad \swarrow {\scriptstyle \mathrm{id}_a \otimes l_b} & \\
& a \otimes b &
\end{array} , \tag{A.1.2}$$

called the *triangle identity* (for $a, b \in \mathrm{ob}(\mathcal{M}_0)$).

Definition A.1.2. (Monoidal functor)
A functor $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories $(\mathcal{C}, \otimes_\mathcal{C}, I_\mathcal{C})$ and $(\mathcal{D}, \otimes_\mathcal{D}, I_\mathcal{D})$ is said to be *monoidal* if it comes equipped with

- a natural transformation $\phi_{a,b} : F(a) \otimes_\mathcal{D} F(b) \to F(a \otimes_\mathcal{C} b)$ and
- a morphism $\phi : I_\mathcal{D} \to F(I_\mathcal{C})$,

subject to the following conditions:

- (associativity) for all $a, b, c \in \mathrm{ob}(\mathcal{C})$, the following diagram commutes:

$$\begin{CD}
(F(a) \otimes_\mathcal{D} F(b)) \otimes_\mathcal{D} F(c) @>{\phi_{a,b} \otimes_\mathcal{D} \mathrm{id}_\mathcal{D}}>> F(a \otimes_\mathcal{C} b) \otimes_\mathcal{D} F(c)\\
@V{\alpha_{F(a),F(b),F(c)}}VV @VV{\phi_{a \otimes_\mathcal{C} b, c}}V\\
F(a) \otimes_\mathcal{D} (F(b) \otimes_\mathcal{D} F(c)) @. F((a \otimes_\mathcal{C} b) \otimes_\mathcal{C} c)\\
@V{\mathrm{id}_\mathcal{D} \otimes_\mathcal{D} \phi_{b,c}}VV @VV{F(\alpha_{a,b,c})}V\\
F(a) \otimes_\mathcal{D} F(b \otimes_\mathcal{C} c) @>{\phi_{a, b \otimes_\mathcal{C} c}}>> F(a \otimes_\mathcal{C} (b \otimes_\mathcal{C} c))
\end{CD} , \tag{A.1.3}$$

- (unitality) for all $a \in \mathrm{ob}(\mathcal{C})$ the following diagram commutes

$$\begin{CD}
I_\mathcal{D} \otimes_\mathcal{D} F(a) @<{l_{F(a)}}<< F(a)\\
@V{\phi \otimes_\mathcal{D} \mathrm{id}_\mathcal{D}}VV @VV{F(l_a)}V\\
F(I_\mathcal{C}) \otimes_\mathcal{D} F(a) @>{\phi_{I_\mathcal{C}, a}}>> F(I_\mathcal{C} \otimes_\mathcal{D} a)
\end{CD} , \tag{A.1.4}$$

  and similarly for the right unitors r_*.

Definition A.1.3. (Symmetric monoidal category)
A *symmetric monoidal category* $\mathcal{C}$ consists of a monoidal category $\mathcal{M}$ together with a natural isomorphism $B_{a,b} : a \otimes b \to b \otimes a$ (called braiding) obeying

$$B_{b,a} B_{a,b} = \mathrm{id}_{a \otimes b},$$

such that the following diagram commutes (this is called the *hexagon identity*):

$$\begin{CD}
(a \otimes b) \otimes c @>{\alpha_{a,b,c}}>> a \otimes (b \otimes c) @>{B_{a, b \otimes c}}>> (b \otimes c) \otimes a\\
@V{B_{a,b} \otimes \mathrm{id}}VV @. @VV{\alpha_{b,c,a}}V\\
(b \otimes a) \otimes c @>{\alpha_{b,a,c}}>> b \otimes (a \otimes c) @>{\mathrm{id} \otimes B_{a,c}}>> b \otimes (c \otimes a)
\end{CD} . \tag{A.1.5}$$

Definition A.1.4. (Symmetric monoidal functor)
A monoidal functor F between symmetric monoidal categories $\mathcal{C}, \mathcal{D}$ is called *symmetric* if it commutes with the braiding, i.e. if it makes the following diagram commute for any $a, b \in \mathrm{ob}(\mathcal{C})$:

$$\begin{array}{ccc} F(a) \otimes_{\mathcal{D}} F(b) & \xrightarrow{B_{F(a),F(b)}} & F(b) \otimes_{\mathcal{D}} F(a) \\ \downarrow \phi_{a,b} & & \downarrow \phi_{b,a} \\ F(a \otimes_{\mathcal{C}} b) & \xrightarrow{F(B_{a,b})} & F(b \otimes_{\mathcal{C}} a) \end{array} . \tag{A.1.6}$$

A.2 Internal categories

The definition of an *internal category* is very well known and can be found already in [ML71, § XII.1]. Here and in the following subsection on pseudo-categories, we will mostly stick to the definitions found in [ST11] and in [MF06]. Note however that the notations vary.
The rough idea is the following: Part of the data of a category are a *class* of objects and a *class* of morphisms. A *small category* is the special case when the objects and morphisms form actual sets, i.e. when they are objects in the category **Set** of sets and functions. A small category is then the same as an internal category in **Set**. More generally, one could take the objects and morphisms to be objects of other categories, which is formalized in the following definition (which essentially builds a category from scratch):

Definition A.2.1. (Internal category)
Let $\mathcal{A}$ be a category (the $\mathcal{A}$ stands for *ambient*) with pullbacks. A *category* $\mathcal{C}$ *internal to* $\mathcal{A}$ consists of the following data:

- objects $C_0, C_1 \in \mathrm{ob}(\mathcal{A})$ called the *object of objects* resp. *object of morphisms*,
- morphisms $s, t \in \mathrm{mor}(C_1, C_0)$ called the *source* resp. *target morphism*,
- a morphism $e \in \mathrm{mor}(C_0, C_1)$ called the *identity morphism*,
- a morphism $c \in \mathrm{mor}(C_1 \times_{C_0} C_1, C_1)$ called the *composition morphism*, where the pullback $C_1 \times_{C_0} C_1$ is defined by the square

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_1} & C_1 \\ \pi_2 \downarrow & & \downarrow t \\ C_1 & \xrightarrow{s} & C_0 \end{array} , \tag{A.2.1}$$

subject to the following conditions, expressing the usual laws for categories:

- the law specifying source and target of the identity morphism:

$$\begin{array}{ccccc} C_1 & \xleftarrow{e} & C_0 & \xrightarrow{e} & C_1 \\ & \searrow^{s} & \downarrow{\scriptstyle \mathrm{id}} & \swarrow_{t} & \\ & & C_0 & & \end{array} \quad , \tag{A.2.2}$$

- the law specifying source and target of the composition morphism:

$$\begin{array}{ccccc} C_1 & \xleftarrow{\pi_1} & C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \mathrm{c}} & & \downarrow{\scriptstyle t} \\ C_0 & \xleftarrow{\mathrm{s}} & C_1 & \xrightarrow{\mathrm{t}} & C_0 \end{array} \quad , \tag{A.2.3}$$

- the law demanding associativity of the composition of morphisms:

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{\mathrm{id} \times_{C_0} c} & C_1 \times_{C_0} C_1 \\ {\scriptstyle c \times_{C_0} \mathrm{id}}\downarrow & & \downarrow{\scriptstyle c} \\ C_1 \times_{C_0} C_1 & \xrightarrow{\mathrm{c}} & C_1 \end{array} \quad , \tag{A.2.4}$$

- and the law for left and right unitality:

$$\begin{array}{ccccc} C_1 \times_{C_0} C_0 & \xrightarrow{\mathrm{id} \times_{C_0} e} & C_1 \times_{C_0} C_1 & \xleftarrow{e \times_{C_0} \mathrm{id}} & C_0 \times_{C_0} C_1 \\ & \searrow_{\pi_1} & \downarrow{\scriptstyle \mathrm{c}} & \swarrow_{\pi_2} & \\ & & C_1 & & \end{array} \quad . \tag{A.2.5}$$

Now given two categories $\mathcal{C}, \mathcal{D}$ internal to the same ambient category $\mathcal{A}$, one can define:

Definition A.2.2. (Internal functor)
An *internal functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a pair of morphisms $F_0 : C_0 \to D_0$, $F_1 : C_1 \to D_1$ that commute in the obvious way with source and target morphism:

$$s' F_1 = F_0 s \ , \ t' F_1 = F_0 t', \tag{A.2.6}$$

as well as with composition and unit morphism (given by commutative diagrams (A.3.3) with the 2-morphisms discarded).

Note that from here on, s and t denote the source resp. target morphism in $\mathcal{C}$ while s' and t' denote those in $\mathcal{D}$ (and similarly for c, u, etc.). Lastly, given two internal functors $F, G : \mathcal{C} \to \mathcal{D}$, one can define according to [ST11]:

Definition A.2.3. (Internal natural transformation)
An internal natural transformation is a morphism $n : C_0 \to D_1$ that makes the following

diagram commutative:

$$\begin{array}{ccccc} & & C_0 & & \\ & \swarrow^{F_0} & \downarrow n & \searrow^{G_0} & \\ D_0 & \xleftarrow{s'} & D_1 & \xrightarrow{t'} & D_0 \end{array} \quad , \tag{A.2.7}$$

as well as the diagram (A.3.5) considered as a commutative diagram after discarding the 2-morphism ν.

A.3 Pseudo-categories

However, our purposes demand to go a step beyond that and consider not only internal categories in a category but rather *internal categories in a strict 2-category* and then weaken the associativity (A.2.4) and unitality (A.2.5) by demanding them to hold only up to coherent isomorphisms. Those are given by diagrams (2.12) and (2.14) in [ST11] but for convenience we replicate them here. There should be invertible natural transformations λ and ρ (*left* resp. *right unitor*) between the functors in (A.2.5), i.e.

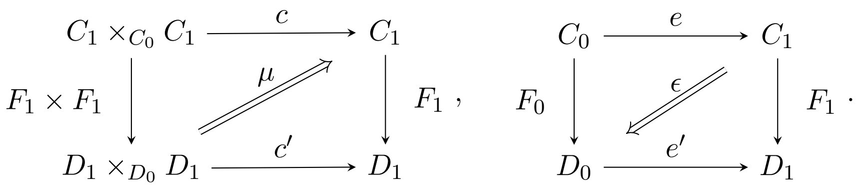
$$\begin{array}{ccccc} C_1 \times_{C_0} C_0 & \xrightarrow{\mathrm{id}\times_{C_0} e} & C_1 \times_{C_0} C_1 & \xleftarrow{e\times_{C_0}\mathrm{id}} & C_0 \times_{C_0} C_1 \\ & {}_{\pi_1}\searrow \overset{\rho}{\Leftarrow} & \downarrow \mathrm{c} & \overset{\lambda}{\Rightarrow} \swarrow_{\pi_2} & \\ & & C_1 & & \end{array} \quad , \tag{A.3.1}$$

as well as an invertible natural transformation α (the *associator*) between the functors in (A.2.4), i.e.

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{\mathrm{id}\times_{C_0} c} & C_1 \times_{C_0} C_1 \\ {\scriptstyle c\times_{C_0}\mathrm{id}}\downarrow & \overset{\alpha}{\nearrow} & \downarrow {\scriptstyle c} \\ C_1 \times_{C_0} C_1 & \xrightarrow{\mathrm{c}} & C_1 \end{array} \quad . \tag{A.3.2}$$

A precise account can be found in [MF06] where these categories are called pseudo-categories. The moral is that the 2-morphisms α, λ, ρ need to satisfy additional coherence properties, among them that the 2-morphisms reduce to identity 2-morphisms when horizontally composed with identity 2-morphisms on source resp. target (similar to (A.3.4)). Moreover, there is a pentagon identity similar to (A.1.1) for the associator α and a triangle identity similar to (A.1.2) for the unitors (see (1.3)-(1.7) in [MF06] for details).

Next we need to find the right notion of a functor F between categories $\mathcal{C}, \mathcal{D}$ internal to the same strict 2-category $\mathcal{A}$ (also called a *pseudo-functor*). It has been carried out in [MF06] that in addition to the morphisms $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ that we have considered before, one needs invertible 2-morphisms to relate composition and identity morphisms in the categories, i.e. 2-isomorphisms μ, ϵ in the following sense:

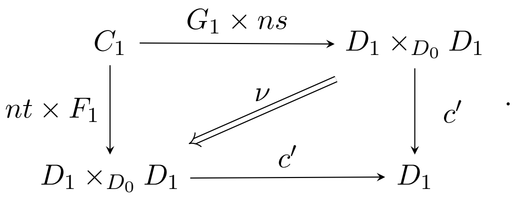


(A.3.3)

The quadruple $(F_0, F_1, \mu, \epsilon)$ again has to satisfy a set of coherence conditions, see (2.2)-(2.6) in [MF06]. In particular, the 1-morphisms have to commute with source and target morphisms as before (A.2.6), while the 2-morphisms again reduce to identity 2-morphisms when horizontally composed with the identity 2-morphisms on source or target morphism, i.e.

$$\begin{aligned} \mathrm{id}_{s'} \circ \mu = \mathrm{id}_{F_0 s \pi_2} \ , \ \mathrm{id}_{t'} \circ \mu = \mathrm{id}_{F_0 t \pi_1}, \\ \mathrm{id}_{s'} \circ \epsilon = \mathrm{id}_{F_0} \ , \ \mathrm{id}_{t'} \circ \epsilon = \mathrm{id}_{F_0}. \end{aligned} \quad \text{(A.3.4)}$$

In addition, there are three more conditions on the composition of 2-morphisms as given by the commutative diagrams (2.5) and (2.6) in [MF06].
Lastly, the notion of a *pseudo-natural transformation* between pseudo-functors $F, G : \mathcal{C} \to \mathcal{D}$ remains to be defined. Again in [MF06] it is shown that in addition to the morphism $n : C_0 \to D_1$ that has already been considered for internal natural transformations, one needs the following 2-isomorphism ν:



(A.3.5)

The diagram (A.2.7) is demanded to remain commutative, while there are additional coherence restrictions on ν ((3.5)-(3.6) in [MF06]). Note that we drop the condition that ν be invertible ((3.4) in [MF06]).

Appendix B

Supermathematics

We use this appendix to review some basic notions of supermathematics that can be found e.g. in [DM99]. The main idea is that one moves from vector spaces, Lie algebras, manifolds, groups, etc. to their respective super versions by including odd degrees of freedom. For example, the sheaf of smooth functions on a manifold is enriched by an exterior algebra (that can be thought of, in a sense, as functions that *anti-commute*).

B.1 Supermanifolds

Let from now on k denote a ground field of characteristic zero.

Definition B.1.1. (Super vector spaces and algebras)

1. A *super vector space* is a $\mathbb{Z}/2$-graded vector space $V = V_0 \oplus V_1$. A homogeneous element $v \in V_0$ (resp. V_1) is called *even* resp. (*odd*), its parity denoted $|v| \in \mathbb{Z}/2$.

2. A *morphism* of super vector spaces V, W is a linear map from V to W that preserves the $\mathbb{Z}/2$-grading.

3. There is a symmetric monoidal category $\mathbf{SVect}_k$ whose objects are super vector spaces over k, the morphisms are as above, the monoidal structure is given by the $\mathbb{Z}/2$-graded tensor product

$$(V \otimes W)_k = \bigoplus_{i+j=k} (V_i \otimes W_j) \, ,$$

   and the braiding isomorphism (on homogeneous elements) by

$$\begin{aligned} B_{V,W} : V \otimes W &\longrightarrow W \otimes V, \\ v \otimes w &\longmapsto (-1)^{|v| \cdot |w|} w \otimes v. \end{aligned}$$

4. There is a *parity reversing functor* Π defined by $(\Pi V)_i := V_{1-i}, i = 0, 1$.

5. For $d_i = \dim V_i < \infty$, the *dimension* of V is the pair (d_0, d_1), usually denoted as $d_0|d_1$.

6. A *super algebra* over k is a super vector space A together with a morphism $m : A \otimes A \to A$, called multiplication. We denote $xy := x \cdot y := m(x \otimes y)$ for $x, y \in A$. Note that by definition $|xy| = |x| + |y|$. We will furthermore require A to be associative (i.e. $x(yz) = (xy)z \ \ \forall x, y, z \in A$) and unital (i.e. it contains an even element 1 such that $1x = x1 = x \ \ \forall x \in A$).

7. A super algebra A is called *commutative*, if $xy = (-1)^{|x| \cdot |y|} yx$ for all homogeneous elements $x, y \in A$.

Definition B.1.2. (Super manifolds)

1. Let $\mathcal{C}^\infty$ be the sheaf of smooth functions on k^p and $\mathcal{C}^\infty[\theta_1, \dots, \theta_q]$ be the sheaf of commutative super k-algebras, freely generated over $\mathcal{C}^\infty$ by odd quantities θ_i. The space $k^{p|q}$ is defined to be the topological space k^p endowed with the sheaf $\mathcal{C}^\infty[\theta_1, \dots, \theta_q]$.

2. A *super manifold* M of dimension $p|q$ is a topological space $|M|$ together with a sheaf $\mathcal{O}_M$ of super k-algebras, which is locally isomorphic to $k^{p|q}$. Abusing notation, the global sections of $\mathcal{O}_M$ are called *functions* on M, which form an algebra $C^\infty(M)$.

3. There is a nilpotent ideal J of $\mathcal{O}_M$ generated by odd functions and $(|M|, \mathcal{O}_M/J)$ is locally isomorphic to $(k^p, \mathcal{C}^\infty)$ (this is clear for $M = k^p$ and then holds by definition for general M). There is thus an underlying smooth p-manifold M_{red} of M, the *reduced* manifold. Topological notions need to be interpreted in terms of this reduced manifold, e.g. an open subset $U \subset M$ is an open subset $|U| \subset |M|$ together with the restriction of the structure sheaf $\mathcal{O}_M$ to $|U|$.

4. *Morphisms* of super manifolds are defined as morphisms of ringed spaces, i.e. a morphism $f : M \to N$ consists of a morphism of topological spaces $|f| : |M| \to |N|$ together with a morphism of sheaves of super k-algebras $|f|^*\mathcal{O}_N \to \mathcal{O}_M$.

5. Super manifolds then form a symmetric monoidal category under disjoint union. We denote this category by **SMan** for $k = \mathbb{R}$ and by **csM** for $k = \mathbb{C}$.

B.2 Super Lie algebras

A *super Lie group* is a group object in the category **csM** (or **SMan**) (just like an ordinary Lie group is a group object in **Man**). Moreover, a *super Lie algebra* is formed by the left invariant vector fields of a super Lie group, completely analogous to the non-super case. Spelled out, a super Lie algebra can be defined as follows:

Definition B.2.1. A *super Lie algebra* over a field k (of characterisic zero) consists of

1. a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathbf{SVect}_k$;

2. a bilinear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, called the *Lie superbracket*, that is *super skew-symmetric*: on homogeneous elements x, y it satisfies

$$[x, y] = -(-1)^{|x|\cdot|y|}[y, x] \ ;$$

3. such that the *super Jacobi identity* holds:

$$\Big[x, [y, z]\Big] = \Big[[x, y], z\Big] + (-1)^{|x|\cdot|y|}\Big[y, [x, z]\Big]$$

   for homogeneous elements x, y, z.

For even elements of the super Lie algebra, these conditions reduce to the ones known from ordinary Lie algebras. In fact, there is a second characterization of super Lie algebras that makes direct contact to the non-super case:

Remark B.2.2. Equivalently, a super Lie algebra is a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that

1. $\mathfrak{g}_0$ is a Lie algebra;

2. $\mathfrak{g}_1$ is a linear representation of $\mathfrak{g}_0$;

3. there exists a $\mathfrak{g}_0$-equivariant linear map $\{-,-\} : \mathrm{Sym}^2\mathfrak{g}_1 \to \mathfrak{g}_0$ such that for all $x, y, z \in \mathfrak{g}_1$ the following holds:

$$\{x, y\}[z] + \{y, z\}[x] + \{z, x\}[y] = 0.$$

Remark B.2.3. In this spirit, a super Lie algebra can be viewed as an extension of an ordinary Lie algebra and in many ways notions from ordinary Lie algebras extend to super Lie algebras (while others may exist only in the ordinary or only in the super case). For example, every finite-dimensional super Lie algebra admits a finite-dimensional faithful representation (which is also true for ordinary finite-dimensional Lie algebras). On the other hand, the Levi decomposition (writing a finite-dimensional Lie algebra as semi-direct product of a solvable ideal (the radical) and a semi-simple Lie subalgebra (the Levi subalgebra)) does not hold for all finite-dimensional super Lie algebras. For example, it fails for $\mathfrak{sl}(m|m)$, $m \geq 2$ which leads us to the first of two "basic" examples of super Lie algebras (where we follow [CW12]):

Example B.2.4. ($\mathfrak{sl}(m|n)$)
Let $V = V_0 \oplus V_1$ be a super vector space such that $\mathrm{End}(V)$ is a superalgebra. One can then equip $\mathrm{End}(V)$ with a super Lie bracket that is given by

$$[x, y] = x \cdot y - (-1)^{|x| \cdot |y|} y \cdot x$$

on homogeneous elements and extended bilinearly. This forms a super Lie algebra denoted by $\mathfrak{gl}(V)$ or by $\mathfrak{gl}(m|n)$ if $V = \mathbb{C}^{m|n}$, which we will assume from now on. Fixing bases for V_0 and V_1 and combining them gives a homogeneous base for V, with respect to which the elements of $\mathrm{End}(m|n)$ and $\mathfrak{gl}(m|n)$ are $(m+n) \times (m+n)$-matrices of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{B.2.1}$$

where A, B, C, D are respectively $m \times m$, $m \times n$, $n \times m$, $n \times n$ matrices. The even subalgebra $\mathfrak{gl}(m|n)_0$ consists of such matrices with $B = 0 = C$ and the odd subspace $\mathfrak{gl}(m|n)_1$ of those with $A = 0 = D$. It is then easy to define the *supertrace* str on $\mathfrak{gl}(m|n)$ via $\mathrm{str}(M) := \mathrm{tr}(A) - \mathrm{tr}(D)$, with tr denoting the usual trace on square matrices. str has the important property that $\mathrm{str}[M, M'] = 0$ for $M, M' \in \mathfrak{gl}(m|n)$ which means that

$$\mathfrak{sl}(m|n) := \{M \in \mathfrak{gl}(m|n) | \ \mathrm{str}(M) = 0\}$$

carries the structure of a super Lie algebra. It is called the *special linear Lie superalgebra* and has the following properties:

- $[\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)] = \mathfrak{sl}(m|n)$;

- $\mathfrak{sl}(m|n) = \mathfrak{sl}(n|m)$;

- $\mathfrak{sl}(m|n)$ is simple for $m \neq n$;

- the center of $\mathfrak{sl}(m|m)$ consists of multiples of the identity $\mathbb{1}_{m|m}$ and the quotient $\mathfrak{sl}(m|m)/\mathbb{1}_{m|m}$ is simple for $m \geq 2$.

This explains why $\mathfrak{sl}(m|n)$ appears in the first two rows of table B.1 the way it does.

Example B.2.5. ($\mathfrak{osp}(m|2n)$)
Another "basic" example of super Lie algebras are the *ortho-symplectic* ones. Fix a super vector space $V = V_0 \oplus V_1$. A bilinear form $B : V \times V \to V$ is called

- $\left\{ \begin{array}{c} \textit{even} \\ \textit{odd} \end{array} \right\}$ if $B(V_i, V_j) = 0$ for $i + j = \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\}$ (mod 2);
- *supersymmetric* if it is even, $B|_{V_0 \times V_0}$ is symmetric and $B|_{V_1 \times V_1}$ skew-symmetric;
- *skew-supersymmetric* if it is even, $B|_{V_1 \times V_1}$ is symmetric and $B|_{V_0 \times V_0}$ skew-symmetric.

Let B be a non-degenerate supersymmetric bilinear form on V (which means that in particular $\dim(V_1)$ is even). For $k = 0, 1$ we define

$$\begin{aligned} \mathfrak{osp}(V)_k &:= \left\{ M \in \mathfrak{gl}(V)_k \,\middle|\, \begin{array}{c} B(Mx, y) = -(-1)^{k \cdot |x|} B(x, My) \\ \forall x, y \in V, x \text{ homogeneous} \end{array} \right\}, \\ \mathfrak{osp}(V) &:= \mathfrak{osp}(V)_0 \oplus \mathfrak{osp}(V)_1. \end{aligned}$$

$\mathfrak{osp}(V)$ is indeed a super Lie algebra with even part $\mathfrak{osp}(V)_0 = \mathfrak{so}(V_0) \oplus \mathfrak{sp}(V_1)$. Hence, it is called the *ortho-symplectic super Lie algebra* and we write $\mathfrak{osp}(m|2n)$ when $V = \mathbb{C}^{m|2n}$. One can similarly define the super Lie algebra $\mathfrak{spo}(2n|m)$ to be the subalgebra of $\mathfrak{gl}(2n|m)$ that preserves a non-degenerate *skew-supersymmetric* bilinear form but it becomes redundant for us due to an isomorphism

$$\mathfrak{spo}(2n|m) \cong \mathfrak{osp}(m|2n).$$

A very full account on ortho-symplectic superalgebras can be found in [Far84].

Kac ([Kac77a], [Kac77b]) was able to completely classify finite dimensional super Lie algebras that are simple and over a field of characteristic zero. The super Lie algebras in that list are primarily of two types: either of *classical type* (for which the action of the even part on the odd part is completely reducible) or of *Cartan type* (for which it is not completely reducible). For the classical type there is again a distinction into *basic* ones and ones of *queer* or *strange type*, and lastly one differs between basic classical super Lie algebras of type I and type II depending on whether the action of the even part on the odd part is irreducible or not. The list of the basic classical super Lie algebras can be found in table B.1.

We are now interested in how Poincaré and conformal algebras extend to super Lie algebras. Let us first recall the notions of these Lie algebras:

1. $\mathbb{M}^{p,q}$ is the real affine space with an underlying $(p+q)$-dimensional vector space $\mathbb{R}^{p,q}$ and a translationally invariant pseudo-Riemannian metric of signature (p, q)[3]. In the case $q = 1$ it is called the $(p+1)$-dimensional *Minkowski space.*

2. The d-dimensional Poincaré group is the spin double-covering of the connected component of the identity of the isometry group of $\mathbb{M}^{d-1,1}$:

$$\mathrm{ISO}(d-1,1) \cong \mathrm{Spin}(d-1,1) \ltimes \mathbb{R}^{d-1,1}; \tag{B.2.2}$$

[1]There is an isomorphism $A(m, m) = \mathfrak{sl}(m+1|m+1)/\mathbb{C}\mathbb{1}_{m+1|m+1}$.

[2]Here, the action of $\mathcal{S}_3$ on $\mathbb{C}\backslash\{0, -1\}$ is generated by $\alpha \mapsto \frac{1}{\alpha}$ and by $\alpha \mapsto -1 - \alpha$, see [CW12] for details.

[3]i.e. the metric can locally be brought into the form $\mathrm{diag}(\underbrace{+1, \ldots, +1}_{p}, \underbrace{-1, \ldots, -1}_{q})$.

$\mathfrak{g}$	Parameters	$\mathfrak{g}_0$
$A(m,n) = \mathfrak{sl}(m+1\|n+1)$	$m > n \geq 0$, $(m,n) \neq (1,0)$	$A_m \oplus A_n \oplus \mathbb{C}$
$A(m,m)^1$	$m \geq 1$	$A_m \oplus A_m$
$B(m,n) = \mathfrak{osp}(2m+1\|2n)$	$m \geq 0, n \geq 1$	$B_m \oplus C_n$
$C(n) = \mathfrak{osp}(2\|2n-2)$	$n \geq 2$	$C_{n-1} \oplus \mathbb{C}$
$D(m,n) = \mathfrak{osp}(2m\|2n)$	$m \geq 2, n \geq 1$	$D_m \oplus C_n$
$G(3)$		$G_2 \oplus A_1$
$F(4)$		$B_3 \oplus A_1$
$D(2,1;\alpha)$	$\alpha \in (\mathbb{C}\backslash\{0,-1\})/\mathbb{S}_3{}^2$	$A_1 \oplus A_1 \oplus A_1$

TABLE B.1: The list of basic classical super Lie algebras. The A and C series are of type I, the others are of type II. The structures of $\mathfrak{g}_1$ as $\mathfrak{g}_0$-modules are collected in [Far84].

3. The d-dimensional Poincaré algebra is the Lie algebra $\mathfrak{iso}(d-1,1)$ of the d-dimensional Poincaré group $\mathrm{ISO}(d-1,1)$.

4. Let $p+q > 2$. The conformal group $\mathrm{CO}(p,q)$ is the identity component in the group of conformal diffeomorphisms of the conformal compactification (for a definition see [Sch97], chapter 2.1) of $\mathbb{M}^{p,q}$. It is isomorphic to $\mathrm{SO}(p+1,q+1)$ (or to $\mathrm{SO}(p+1,q+1)/\{\pm\mathbb{1}\}$ if $-\mathbb{1}$ is in the connected component of the identity) according to theorem 2.9 in [Sch97].

5. The conformal algebra $\mathfrak{co}(p,q)$ is the Lie algebra to the Lie group $\mathrm{CO}(p,q)$.

Let us now recall how the super-analogues of the first three on this list are defined (which is mostly adopted from [DF99b]):

Definition B.2.6. (Super Minkowski space, super Poincaré group and -algebra)

1. $\mathbb{M}^{p,q|S}$ consists of $\mathbb{M}^{p,q}$, a real spinorial representation S of $\mathrm{Spin}(p,q)$, a positive cone C of timelike vectors in $\mathbb{R}^{p,q}$ (that is vectors v for which $v \cdot v < 0$) and a symmetric morphism Γ of representations of $\mathrm{Spin}(p,q)$:

$$\Gamma : S^* \otimes S^* \to \mathbb{R}^{p,q}$$

   such that $\Gamma(s^*, s^*) \in \bar{C} \quad \forall s^* \in S^*$ and $\Gamma(s^*, s^*) = 0$ only for $s^* = 0$ (mimicking positive definiteness). Such a Γ exists for every choice of representation S and is unique up to automorphisms of S. In particular, $\mathbb{M}^{p,q|S}$ can be considered as a super affine space with even part $\mathbb{M}^{p,q}$ and odd part ΠS^*, the vector space underlying S^*.
   If S is the sum of k irreducible real representations of $\mathrm{Spin}(p,q)$, one speaks of $\mathcal{N} = k$ supersymmetry and thus eases the notation to $\mathbb{M}^{p,q|k}$. This breaks down in dimensions $p+q \equiv 2 \bmod 4$ where two inequivalent real irreducible representations S^+ and S^- exist. Thus, one will speak of $\mathcal{N} = (k^+, k^-)$ supersymmetry and denote the superspace by $\mathbb{M}^{p,q|(k^+,k^-)}$ (here $k^\pm$ is the multiplicity with which $S^\pm$ appears in the decomposition of S into irreducible representations).

2. Let $q = 1$ and fix a representation S of $\mathrm{Spin}(d-1,1)$. The *super Poincaré algebra* $\mathfrak{siso}_S(d-1,1)$ is the super Lie algebra with even part $\mathfrak{iso}(d-1,1)$ and whose odd part given by ΠS^*. The graded Lie bracket is given as follows:

- The super Lie bracket between even elements is simply the Lie bracket of the Lie algebra $\mathfrak{iso}(d-1,1)$.
- The Lie bracket between two odd elements is given by a multiple of Γ, i.e. $[s_1, s_2] := -2 \cdot \Gamma(s_1, s_2)$.
- The super Lie bracket of an odd element with an element from $\mathfrak{so}(d-1,1)$ is given by the action, the super Lie bracket of an odd element with one from $\mathbb{R}^{d-1,1}$ is trivial.

This is indeed a super Lie algebra thanks to the symmetry and $\mathrm{Spin}(d-1,1)$ equivariance of Γ: the symmetry amounts to the graded skew-symmetry on odd elements, while Γ being a morphism of $\mathrm{Spin}(d-1,1)$-representations yields the non-trivial Jacobi identity: for $s_1, s_2 \in S^*$ and $x \in \mathfrak{so}(d-1,1)$ the following holds by definition of Γ:

$$\Big[[x,s_1],s_2\Big] + \Big[s_1,[x,s_2]\Big] = -2\cdot\Gamma\Big([x,s_1],s_2\Big) + -2\cdot\Gamma\Big(s_1,[x,s_2]\Big)$$
$$= -2\cdot\Big[x,\Gamma(s_1,s_2)\Big] = \Big[x,[s_1,s_2]\Big].$$

The other conditions are trivially satisfied. Note that in particular $[[s_1, s_2], s_3]$ always vanishes for $s_i \in S^*$.

3. The *super Poincaré group* $\mathrm{SISO}_S(d-1,1)$ is the unique (up to isomorphism) simply connected super Lie group, for which there is an isomorphism between its Lie algebra of left-invariant vector fields and $\mathfrak{siso}_S(d-1,1)$. The action of this group is called a *supersymmetry* in physics. Note that there is an isomorphism

$$\mathrm{SISO}_S(d-1,1) \cong \mathrm{Spin}(d-1,1) \ltimes \mathbb{R}^{d-1,1|S},$$

where $\mathbb{R}^{d-1,1|S}$ is identified with the super Lie group of super translations on the super affine space $\mathbb{M}^{d-1,1|S}$ (similar to how $\mathbb{R}^{d-1,1}$ is identified with the group of translations on the underlying affine space in B.2.2).

Remark B.2.7. The connected component of the subgroup of outer automorphisms of the super Poincaré group which fix the Poincaré subgroup is called the *R-symmetry group* and similarly there is an *R-symmetry algebra*. These are frequently absorbed into the definition of the super Poincaré group resp. algebra.

Remark B.2.8. Rather than to give a full definition of a superconformal algebra $\mathfrak{sco}_S(d-1,1)$, we settle for pointing out three properties that it should naturally fulfill:

- $\mathfrak{sco}_S(d-1,1)$ acts as infinitesimal transformations on $\mathbb{M}^{d-1,1|S}$;
- this action extends the infinitesimal action of $\mathfrak{siso}_S(d-1,1)$ on $\mathbb{M}^{d-1,1|S}$;
- restricting this action to the action of the even part on ordinary Minkowski space $\mathbb{M}^{d-1,1}$ is an extension of the action of $\mathfrak{co}(d-1,1)$ on $\mathbb{M}^{d-1,1}$.

In other words, the even part needs to possess $\mathfrak{so}(d,2)$ as subalgebra with a spinorial representation on the odd part. Shnider has shown in [Shn88] that the existence of a superconformal algebra in d dimensions imposes the existence of a simple one. Thus, one would only need to check the list of simple super Lie algebras (of which we have only given a portion in table B.1) and look for the possible subalgebras in even degree and their action on the odd subspace. As it turns out, the superconformal algebras

d	$\mathcal{N}$	superconformal algebra	R-symmetry
3	$2k+1$	$B(k,2) \cong \mathfrak{osp}(2k+1\|4)$	$\mathrm{SO}(2k+1)$
3	$2k$	$D(k,2) \cong \mathfrak{osp}(2k\|4)$	$\mathrm{SO}(2k)$
4	$k+1$	$A(3,k) \cong \mathfrak{sl}(4\|k+1)$	$\mathrm{U}(k+1)$
5	1	$F(4)$	$\mathrm{SO}(3)$
6	$(k,0)$	$D(4,k) \cong \mathfrak{osp}(8\|2k)$	$\mathrm{Sp}(k)$

TABLE B.2: The list of superconformal algebras $\mathfrak{sco}_S(d-1,1)$ extending the super Poincaré algebra $\mathfrak{siso}_S(d-1,1)$. Here, $\mathcal{N}$ denotes the number of irreducible representations in the decomposition of the odd part of the super Lie algebra.

are governed by special isomorphisms of ordinary Lie algebras (see e.g. [Min98, § 4.2]) which only exist in low dimensions. In fact, there are no superconformal extensions of the super Poincaré algebra for $d > 6$. For $3 \leq d \leq 6$, the list of superconformal algebras can be found in table B.2.

B.3 Representations of the super Poincaré algebra

In this section we want to recap the representation theory of super Poincaré algebras. Because these extend regular Poincaré algebras, precomposing a representation ρ with the inclusion map

$$\mathfrak{iso}(d-1,1) \hookrightarrow \mathfrak{siso}_S(d-1,1) \xrightarrow{\rho} \mathfrak{end}(V)$$

gives a representation of the Poincaré algebra. We will see that an irreducible representation of a super Poincaré algebra is fully reducible as a representation of the ordinary Poincaré algebra and that moreover the number of irreducible representations it decomposes into varies: the minimal case corresponds to massless particles or so called *BPS states*, which play a crucial role in the physics of supersymmetry. We will be rather concrete in our calculations and will always use Einstein notation, i.e.

- upper (resp. lower) indices represent contravariant (resp. covariant) tensor indices;
- if an index appears twice in a single term (and is not otherwise defined), it is a summation index, e.g. $P_\mu P^\mu := \sum_{\mu=0}^{d-1} P_\mu \cdot P^\mu$.

First off, note that we will restrict our attention to the case $d = 4$ in this section. Recall that by the *Coleman-Mandula-Theorem*, for any *interacting* QFT satisfying "reasonable" *physical* assumptions (locality, causality, positivity of energy, finiteness of number of particles, mass gap) the symmetry Lie algebra must be a direct sum of the Poincaré algebra and an internal Lie algebra. Let us denote the generators of the Poincaré algebra $\mathfrak{iso}(d-1,1) = \mathbb{R}^{d-1,1} \rtimes \mathfrak{so}(d-1,1)$ by P_μ (translations) and $M_{\mu\nu}$ (Lorentz transformations), the generators of the internal Lie algebra by B_l and the Minkowski metric by $\eta_{\mu\nu}$, then the commutation relations must take the following form:

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = i\left(M_{\mu\sigma}\eta_{\nu\rho} + M_{\nu\rho}\eta_{\mu\sigma} - M_{\mu\rho}\eta_{\nu\sigma} - M_{\nu\sigma}\eta_{\mu\rho}\right), \\
&[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\rho] = i\left(P_\mu\eta_{\nu\rho} - P_\nu\eta_{\mu\rho}\right), \\
&[B_l, B_m] = i f_{lm}^n B_n, \quad [P_\mu, B_l] = 0 = [M_{\mu\nu}, B_l].
\end{aligned} \tag{B.3.1}$$

An analogue characterization for the super extensions is given by the *Haag - Lopuszanski - Sohnius Theorem*: under similar assumptions, the only possible super Lie algebra symmetry of an S-matrix is a direct sum of the super Poincaré Lie algebra and another super Lie algebra of internal symmetries. In particular, denoting the generators of the odd part by Q_α^I and $\bar{Q}_{\dot{\alpha}}^I = (Q_\alpha^I)^\dagger$ (here, $I = 1, ..., \mathcal{N}$ counts the number of generators, $\alpha, \dot{\alpha} = 1, 2$ are the spinor indices and the dot over the index α is just for notational purposes), the super Lie bracket relations of the generators consist of those from B.3.1 and the following list (where for notational purposes $\{A, B\}$ denotes the super Lie bracket between odd generators):

$$[P_\mu, Q_\alpha^I] = 0 = [P_\mu, \bar{Q}_{\dot{\alpha}}^I], \tag{B.3.2}$$

$$[M_{\mu\nu}, Q_\alpha^I] = i\,(\sigma_{\mu\nu})_\alpha^{\ \beta}\, Q_\beta^I, \quad [M_{\mu\nu}, Q^{I\dot{\alpha}}] = i\,(\bar{\sigma}_{\mu\nu})_{\ \dot{\beta}}^{\dot{\alpha}}\, Q^{I\dot{\beta}}, \tag{B.3.3}$$

$$\left\{Q_\alpha^I, \bar{Q}^{J\dot{\beta}}\right\} = 2\,(\sigma^\mu)_\alpha^{\ \dot{\beta}}\, P_\mu \delta^{IJ}, \tag{B.3.4}$$

$$[Q_\alpha^I, B_l] = (b_l)_{\ J}^{I}\, Q_\alpha^J, \quad [\bar{Q}_{I\dot{\alpha}}, B_l] = -\bar{Q}_{J\dot{\alpha}}\,(b_l)_{\ I}^{J}\,, \tag{B.3.5}$$

$$\left\{Q_\alpha^I, Q_\beta^J\right\} = 2\epsilon_{\alpha\beta} Z^{IJ}, \quad \left\{\bar{Q}_{\dot{\alpha}}^I, \bar{Q}_{\dot{\beta}}^J\right\} = 2\epsilon_{\dot{\alpha}\dot{\beta}}\left(Z^{IJ}\right)^*. \tag{B.3.6}$$

Here, σ^μ denotes a 4-vector of 2×2 matrices, where σ^0 is the identity matrix and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

denote the Pauli matrices, with the 2-index Pauli matrices defined as

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} := \frac{1}{4}\left(\sigma^\mu_{\alpha\dot{\gamma}}\,(\bar{\sigma}^\nu)^{\dot{\gamma}\beta} - \sigma^\nu_{\alpha\dot{\gamma}}\,(\bar{\sigma}^\mu)^{\dot{\gamma}\beta}\right) \tag{B.3.7}$$

where ϵ is the Levi-Civita tensor (a.k.a. the totally antisymmetric tensor) and

$$(\bar{\sigma}^\mu)^{\alpha\dot{\alpha}} := \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\,(\sigma^\mu)_{\beta\dot{\beta}} \tag{B.3.8}$$

is the conjugate to σ^μ. Similarly, one defines $\bar{\sigma}_{\mu\nu}$. The tensors b_l that appear in (B.3.5) are structure constants that depend on the R-symmetry algebra and Z^{IJ} from equation (B.3.6) are antisymmetric (i.e. $Z^{IJ} = -Z^{JI}$) central charges, meaning that they commute with all operators. We skip the derivations of the relations (B.3.2)-(B.3.6), they can be found e.g. in [QKS10], p. 22ff. However, let us note that the procedure for each of those equations involves an educated guess followed by exploitation of the super Jacobi identities.

There are some immediate consequences arising from the super Lie bracket relations. Firstly, every state $|\psi\rangle$ has positive energy, i.e.

$$\langle\psi|\, P_0\, |\psi\rangle \geq 0 \tag{B.3.9}$$

where we used the Dirac bra-ket notation. The proof is straightforward and can be found in [Ber15], but let us quickly recall it: for any choice of α and $\dot{\alpha}$, equation (B.3.4) grants

$$\begin{aligned}
2\sigma^{\mu}_{\alpha\dot{\alpha}} \langle\psi| P_\mu |\psi\rangle &= \langle\psi| \left\{Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}\right\} |\psi\rangle \\
&= \langle\psi| \left(Q^I_\alpha \bar{Q}^I_{\dot{\alpha}} + \bar{Q}^I_{\dot{\alpha}} Q^I_\alpha\right) |\psi\rangle \\
&= \langle\psi| \left(Q^I_\alpha (Q^I_\alpha)^\dagger + (Q^I_\alpha)^\dagger Q^I_\alpha\right) |\psi\rangle \\
&= \left\|Q^I_\alpha\right\|^2 + \left\|(Q^I_\alpha)^\dagger\right\|^2 \\
&\geq 0
\end{aligned} \tag{B.3.10}$$

due to positivity of the Hilbert space. Summing over $\alpha = \dot{\alpha} = 1, 2$ results in a trace term and using $\mathrm{tr}(\sigma^\mu) = 2\delta^{\mu 0}$ leads to the desired

$$4 \langle\psi| P_0 |\psi\rangle \geq 0.$$

Recall that a particle of mass $m > 0$ can be brought to rest in the rest frame, where P_μ takes the form $(m, 0, 0, 0)$ and (B.3.4) thus simplifies to

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\right\} = 2m\ \delta_{\alpha\dot{\beta}}\ \delta^{IJ}. \tag{B.3.11}$$

The matrix (Z^{IJ}) of central charges is antisymmetric and by the spectral theorem can be brought into the form

$$Z = \begin{pmatrix}
0 & Z_1 & & & \dots & 0 & 0 \\
-Z_1 & 0 & & & \dots & 0 & 0 \\
 & & 0 & Z_2 & \dots & 0 & 0 \\
 & & -Z_2 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & Z_k \\
0 & 0 & 0 & 0 & \dots & -Z_k & 0
\end{pmatrix}, \tag{B.3.12}$$

where $2k = \mathcal{N}$ and if $\mathcal{N}$ was odd, there would be an extra row and column of zeroes. This allows the definition of ladder operators $(r = 1, \dots, k)$

$$\begin{aligned}
a^r_\alpha &:= \frac{1}{\sqrt{2}} \left(Q^{2r-1}_\alpha + \epsilon^\beta_\alpha \left(Q^{2k}_\beta\right)^\dagger\right), \\
b^r_\alpha &:= \frac{1}{\sqrt{2}} \left(Q^{2r-1}_\alpha - \epsilon^\beta_\alpha \left(Q^{2k}_\beta\right)^\dagger\right),
\end{aligned} \tag{B.3.13}$$

which are odd and satisfy the following anticommutation relations (all others vanish):

$$\begin{aligned}
\left\{a^r_\alpha, \left(a^s_\beta\right)^\dagger\right\} &= 2\left(m - Z_r\right)\ \delta_{rs}\ \delta_{\alpha\beta}, \\
\left\{b^r_\alpha, \left(b^s_\beta\right)^\dagger\right\} &= 2\left(m + Z_r\right)\ \delta_{rs}\ \delta_{\alpha\beta}.
\end{aligned} \tag{B.3.14}$$

This is remarkable because the positivity of the Hilbert space demands that the anticommutators are non-negative (inserting the equations (B.3.13) leads to squares $QQ^\dagger$ which are positive by the same argument as in (B.3.10)), i.e. there is a bound

$$m \geq |Z_r|\ \ \forall r \in \{1, \dots, k\}. \tag{B.3.15}$$

This is the *BPS-bound* (due to Bogomolnyi, Prasad and Sommerfeld) and states saturating this set of inequalities are known as *BPS-states*.

Before we turn to the actual representation theory, recall that the operator measuring *spin* in the classical setting is the Pauli-Lupanski (pseudo-)vector

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}, \tag{B.3.16}$$

whose square $W^2 := W_\mu W^\mu$ is a Casimir operator of the Poincaré algebra, with the second Casimir given by $P^2 = P_\mu P^\mu$. Now while $C_1 = P^2$ remains a Casimir operator of the super Poincaré algebra (due to (B.3.2)), W^2 fails to be Casimir because

$$[Q_\alpha, W_\mu W^\mu] = 2i\,(\sigma^{\mu\nu})_\alpha^{\ \beta}\,Q_\beta W_\mu P_\nu \neq 0.$$

The correct Casimir in the super setting contains the superspin

$$Y_i = \frac{1}{2}\epsilon_{ijk}M^{jk} - \frac{1}{4m}\bar{Q}\bar{\sigma}_i Q, \tag{B.3.17}$$

with Latin indices $i, j, k \in \{1, 2, 3\}$ (as opposed to Greek indices μ, ν... which take values in $\{0, 1, 2, 3\}$). The second Casimir is

$$C_2 = 2m^4 Y^i Y_i. \tag{B.3.18}$$

The Y_i satisfy the commutation relations of a rotation group $[Y_i, Y_j] = \epsilon_{ijk}Y^k$ and C_2 consequently has eigenvalues $2m^4 y(y + 1)$ with the superspin y. An easy computation shows that a supersymmetry generator Q changes the spin by 1/2, thus relating bosons and fermions.

Now choose a Clifford vacuum, i.e. a state $|\lambda\rangle$ that is annihilated by the annihilation operators a^r_α and b^r_α from (B.3.13). Assuming that none of the central charges saturate the BPS bound (B.3.15), the action of the creation operators $(a^r_\alpha)^\dagger$ and $(b^r_\alpha)^\dagger$ on $|\lambda\rangle$ gives a representation that contains $2^{4k} = 2^{2\mathcal{N}}$ states[4], called a *(long) multiplet*.
On the other hand, assume that $1 \leq t \leq k$ of the central charges saturate the BPS bound. This means that t of the ladder operators become trivial (because a right hand side term must vanish in (B.3.14)) and consequently the multiplet consists of only $2^{2(\mathcal{N}-t)}$ states and is called a *short multiplet*. In the maximal case $t = k$ the multiplet contains only $2^{\mathcal{N}}$ states and is called an *ultra-short multiplet*. Let us finish this section with some general remarks:

- We did not take into account massless representations but those can be seen as limit $m \to 0$ in the following way: to satisfy the BPS bound (B.3.15), all central charges must vanish and consequently the BPS bound is always saturated. As we have just stated, this means that we are dealing with a multiplet with $2^{\mathcal{N}}$ states.

- As we have stated before, the supersymmetry generators change the spin of a state by 1/2. Consequently, the difference of the highest and the lowest spin of states in a $2^{\mathcal{N}}$-multiplet is $\mathcal{N}/2$. This strongly restricts the physical relevant types of supersymmetry because renormalizable interacting local field theories cannot contain particles of spin higher than 1 (or only 2 if one includes gravity). Consequently, one usually only considers supersymmetry with $\mathcal{N} \leq 4$ (or $\mathcal{N} \leq 8$ with gravity). Moreover, under a CPT-transformation the spin is flipped, so a CPT-invariant

[4]because each state is either annihilated by an annihilation operator or not, which gives $2k$ choices

multiplet must necessarily have integer difference between maximal and minimal spin, otherwise the CPT-conjugated multiplet must be added. This leads to an equivalence between the $\mathcal{N} = 3$ and $\mathcal{N} = 4$ cases. Hence, one usually considers only the cases $\mathcal{N} = 1, 2, 4$ (unless gravity is included).

Bibliography

[ADJM12] Evgeny Andriyash, Frederik Denef, Daniel L. Jafferis, and Gregory W. Moore. Wall-crossing from supersymmetric galaxies. *J. High Energy Phys.*, (1):115, 15, 2012.

[AGT10] Luis F. Alday, Davide Gaiotto, and Yuji Tachikawa. Liouville correlation functions from four-dimensional gauge theories. *Lett. Math. Phys.*, 91(2):167–197, 2010.

[Ati88] Michael F. Atiyah. Topological quantum field theory. *Publications Mathématiques de l'IHÉS*, 68:175–186, 1988.

[Bas09] Stéphane Baseilhac. Chern-Simons theory in dimension 3, 2009.

[BD95] John C. Baez and James Dolan. Higher-dimensional Algebra and Topological Quantum Field Theory. *J.Math.Phys.*, 36:6073–6105, 1995.

[Ber11] Julia E. Bergner. Models for (∞, n)-categories and the cobordism hypothesis. In *Mathematical foundations of quantum field theory and perturbative string theory*, volume 83 of *Proc. Sympos. Pure Math.*, pages 17–30. Amer. Math. Soc., Providence, RI, 2011.

[Ber15] Matteo Bertolini. Lectures on Supersymmetry, 2015.

[CFIV92] Sergio Cecotti, Paul Fendley, Ken Intriligator, and Cumrun Vafa. A new supersymmetric index. *Nuclear Phys. B*, 386(2):405–452, 1992.

[Cor10] Vicente Cortés, editor. *Handbook of pseudo-Riemannian geometry and supersymmetry*, volume 16 of *IRMA Lectures in Mathematics and Theoretical Physics*. European Mathematical Society (EMS), Zürich, 2010.

[CV93] Sergio Cecotti and Cumrun Vafa. On classification of $N = 2$ supersymmetric theories. *Comm. Math. Phys.*, 158(3):569–644, 1993.

[CW12] S.J. Cheng and W. Wang. *Dualities and Representations of Lie Superalgebras*. Graduate studies in mathematics. American Mathematical Society, 2012.

[DF99a] Pierre Deligne and Daniel S. Freed. Classical field theory. In *Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997)*, pages 137–225. Amer. Math. Soc., Providence, RI, 1999.

[DF99b] Pierre Deligne and Daniel S. Freed. Supersolutions. In *Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997)*, pages 227–355. Amer. Math. Soc., Providence, RI, 1999.

[DM99] Pierre Deligne and John W. Morgan. Notes on supersymmetry (following Joseph Bernstein). In *Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997)*, pages 41–97. Amer. Math. Soc., Providence, RI, 1999.

[DM11] Frederik Denef and Gregory W. Moore. Split states, entropy enigmas, holes and halos. *J. High Energy Phys.*, (11):129, i, 152, 2011.

[DSPS13] Christopher L. Douglas, Christopher Schommer-Pries, and Noah Snyder. Dualizable tensor categories, 2013.

[Far84] Richard Joseph Farmer. *Orthosymplectic Superalgebras in Mathematics and Physics*. PhD thesis, University of Tasmania, Hobart, TAS, Australia, 1984.

[FG06] Vladimir Fock and Alexander Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.*, (103):1–211, 2006.

[FHLT10] Daniel S. Freed, Michael J. Hopkins, Jacob Lurie, and Constantin Teleman. Topological quantum field theories from compact Lie groups. In *A celebration of the mathematical legacy of Raoul Bott*, volume 50 of *CRM Proc. Lecture Notes*, pages 367–403. Amer. Math. Soc., Providence, RI, 2010.

[Fre99] Daniel S. Freed. Special Kähler manifolds. *Comm. Math. Phys.*, 203(1):31–52, 1999.

[FRS02] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. TFT construction of RCFT correlators. I. Partition functions. *Nuclear Phys. B*, 646(3):353–497, 2002.

[FSS14] Domenico Fiorenza, Hisham Sati, and Urs Schreiber. Multiple M5-branes, string 2-connections, and 7d nonabelian Chern-Simons theory. *Adv. Theor. Math. Phys.*, 18(2):229–321, 2014.

[Gai12] Davide Gaiotto. Surface operators in $\mathcal{N} = 2$ 4d gauge theories. *J. High Energy Phys.*, (11):090, front matter + 24, 2012.

[GMN10] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. Four-dimensional wall-crossing via three-dimensional field theory. *Commun. Math. Phys.*, 299:163–224, 2010.

[GMN12] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. Wall-crossing in coupled 2d-4d systems. *J. High Energy Phys.*, (12):082, front matter + 166, 2012.

[GMN13a] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. Framed BPS States. *Adv. Theor. Math. Phys.*, 17:241–397, 2013.

[GMN13b] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. Spectral networks. *Ann. Henri Poincaré*, 14(7):1643–1731, 2013.

[GMN13c] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. Wall-crossing, Hitchin systems, and the WKB approximation. *Adv. Math.*, 234:239–403, 2013.

[GMN14] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. Spectral networks and snakes. *Ann. Henri Poincaré*, 15(1):61–141, 2014.

[GPS95] Robert Gordon, Anthony John Power, and Ross Street. *Coherence for tricategories*, volume 558. American Mathematical Soc., 1995.

[Gra83] John W. Gray. Review: G. M. Kelly, Basic concepts of enriched category theory. *Bull. Amer. Math. Soc. (N.S.)*, 9(1):102–107, 07 1983.

[Gur06] Nick Gurski. *An Algebraic Theory of Tricategories.* PhD thesis, University of Chicago, Chicago, IL, USA, 2006.

[Hen15] André Henriques. What Chern-Simons theory assigns to a point, 2015.

[Hit92] Nigel Hitchin. Hyper-Kähler manifolds. *Astérisque*, (206):Exp. No. 748, 3, 137–166, 1992. Séminaire Bourbaki, Vol. 1991/92.

[HKLR87] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.

[HN13] L. Hollands and A. Neitzke. Spectral networks and Fenchel-Nielsen coordinates. *arXiv preprint: http://arxiv.org/abs/1312.2979*, December 2013.

[Joy12] Dominic Joyce. On manifolds with corners. In *Advances in geometric analysis*, volume 21 of *Adv. Lect. Math. (ALM)*, pages 225–258. Int. Press, Somerville, MA, 2012.

[Kac77a] V. G. Kac. Lie superalgebras. *Advances in Math.*, 26(1):8–96, 1977.

[Kac77b] V. G. Kac. A sketch of Lie superalgebra theory. *Comm. Math. Phys.*, 53(1):31–64, 1977.

[Kap10] Anton Kapustin. Topological Field Theory, Higher Categories, and Their Applications. pages 2021–2043, 2010.

[Koc04] Joachim Kock. *Frobenius Algebras and 2-D Topological Quantum Field Theories.* Cambridge University Press, 2004.

[KS08] M. Kontsevich and Y. Soibelman. Stability structures, motivic Donaldson-Thomas invariants and cluster transformations, 2008.

[KW07] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric Langlands program. *Commun. Number Theory Phys.*, 1(1):1–236, 2007.

[Lur09] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.

[Mal98a] J. Maldacena. Wilson Loops in Large N Field Theories. *Physical Review Letters*, 80:4859–4862, June 1998.

[Mal98b] Juan Maldacena. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2(2):231–252, 1998.

[MF06] N. Martins-Ferreira. Pseudo-categories. *J. Homotopy Relat. Struct.*, 1(1):47–78, 2006.

[Min98] Shiraz Minwalla. Restrictions imposed by Superconformal Invariance On Quantum Field Theories. *Adv. Theor. Math. Phys.*, 2(4):783–851, 1998.

[ML71] Saunders Mac Lane. *Categories for the Working Mathematician.* Number 5 in Graduate Texts in Mathematics. Springer-Verlag, 1971.

[Moo12] Gregory Moore. Felix Klein lecture notes: Applications of the six-dimensional (2,0) theory to physical mathematics., 2012.

[Nak15] Hiraku Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, I. *arXiv preprint:* `http://arxiv.org/abs/1503.03676`, 2015.

[QKS10] Fernando Quevedo, Sven Krippendorf, and Oliver Schlotterer. Cambridge lectures on supersymmetry and extra dimensions, 2010.

[RT91] Nicolai Reshetikhin and Vladimir Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103(3):547–597, 1991.

[Sch97] Martin Schottenloher. *A Mathematical Introduction into Conformal Field Theory.* Lecture Notes in Physics Monographs. Springer, Berlin, 1997.

[Shn88] Steven Shnider. The superconformal algebra in higher dimensions. *Lett. Math. Phys.*, 16(4):377–383, 1988.

[SP09] Christopher John Schommer-Pries. *The classification of two-dimensional extended topological field theories.* ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–University of California, Berkeley.

[ST11] Stephan Stolz and Peter Teichner. Supersymmetric field theories and generalized cohomology. In *Mathematical foundations of quantum field theory and perturbative string theory*, volume 83 of *Proc. Sympos. Pure Math.*, pages 279–340. Amer. Math. Soc., Providence, RI, 2011.

[Str84] Kurt Strebel. *Quadratic differentials*, volume 5 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1984.

[Str90] Andrew Strominger. Special geometry. *Comm. Math. Phys.*, 133(1):163–180, 1990.

[SW94a] N. Seiberg and E. Witten. Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory. *Nuclear Phys. B*, 426(1):19–52, 1994.

[SW94b] N. Seiberg and E. Witten. Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD. *Nuclear Phys. B*, 431(3):484–550, 1994.

[SW97] N. Seiberg and E. Witten. Gauge dynamics and compactification to three dimensions. In *The mathematical beauty of physics (Saclay, 1996)*, volume 24 of *Adv. Ser. Math. Phys.*, pages 333–366. World Sci. Publ., River Edge, NJ, 1997.

[Til13] O. Till. Supersymmetric Gauge Theory, Wall-Crossing and Hyperkähler Geometry. Master's thesis, Chalmers University of Technology, 2013.

[Tri06] Todd Trimble. Notes on tetracategories. *Available as* `math.ucr.edu/home/baez/trimble/tetracategories.html`, 2006.

[TV08] Bertrand Toën and Gabriele Vezzosi. A note on Chern character, loop spaces and derived algebraic geometry, 2008.

[WB92] Julius Wess and Jonathan Bagger. *Supersymmetry and supergravity*. Princeton Series in Physics. Princeton University Press, Princeton, NJ, second edition, 1992.

[Wit88] Edward Witten. Topological quantum field theory. *Comm. Math. Phys.*, 117(3):353–386, 1988.

[Wit89] Edward Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989.

[Wit91] Edward Witten. Introduction to cohomological field theories. *Internat. J. Modern Phys. A*, 6(16):2775–2792, 1991. Topological methods in quantum field theory (Trieste, 1990).

[Wit96] Edward Witten. Some Comments On String Dynamics. In *Strings '95 (Los Angeles, CA, 1995)*, pages 501–523. World Sci. Publ., River Edge, NJ, 1996.